Pointwise Approximation of Coupled Ornstein-Uhlenbeck Processes

Vom Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte

Dissertation

von
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aus Frankfurt am Main

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Tag der Einreichung:	1. Februar 2012
Tag der mündlichen Prüfung:	4. Mai 2012

Darmstadt 2012
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Acknowledgements

I am grateful to my adviser Prof. Dr. Klaus Ritter for his valuable support and helpful suggestions on my work during the last years. My special thanks are to Prof. Dr. Jens Lang for being co-referee of my thesis.
I also thank Prof. Dr. Thomas Müller-Gronbach, Prof. Dr. Andreas Rößler and Dr. Mehdi Slassi for inspiring discussions and comments.
This work was partially supported by the Deutsche Forschungsgemeinschaft.

Abstract

We consider a stochastic evolution equation on the spatial domain $D = (0,1)^d$, driven by an additive nuclear or space-time white noise, so that the solution is given by an infinite-dimensional Ornstein-Uhlenbeck process. We study algorithms that approximate the mild solution of the equation, which takes values in the Hilbert space $H = L_2(D)$, at a fixed point in time. The error of an algorithm is defined by the average distance between the solution and its approximation in H. The cost of an algorithm is defined by the total number of evaluations of one-dimensional components of the driving H-valued Wiener process W at arbitrary time nodes. We construct algorithms with an asymptotically optimal relation between error and cost. Furthermore, we determine the asymptotic behaviour of the corresponding minimal errors. We show how the minimal errors depend on the spatial dimension d, on the smoothing effect of the semigroup generated by the drift term, on the coupling of the infinite-dimensional system of scalar Ornstein-Uhlenbeck processes, which is specified by the diffusion term, and on the decay of the eigenvalues of W in case of nuclear noise. Asymptotic optimality is achieved by drift-implicit Euler-Maruyama schemes together with non-uniform time discretizations. This optimality cannot necessarily be achieved by uniform time discretizations, which are frequently analyzed in the literature. We complement our theoretical results by numerical studies.

Zusammenfassung

Wir betrachten eine stochastische Evolutionsgleichung auf dem räumlichen Bereich $D = (0,1)^d$, getrieben entweder von einem additiven nuklearen oder einem additiven Raum-Zeit weißen Rauschen, so daß die Lösung durch einen unendlichdimensionalen Ornstein-Uhlenbeck-Prozeß gegeben ist. Wir untersuchen Algorithmen zur Approximation der milden Lösung dieser Gleichung, die Werte in dem Hilbertraum $H = L_2(D)$ annimmt, zu einem festen Zeitpunkt. Der Fehler eines Algorithmus ist definiert durch den mittleren Abstand zwischen der Lösung und ihrer Approximation in H. Die Kosten eines Algorithmus sind definiert durch die Gesamtanzahl der Auswertungen der eindimensionalen Komponenten des treibenden H-wertigen Wiener-Prozesses W an beliebigen Zeitpunkten. Wir konstruieren Algorithmen mit einer asymptotischen optimalen Beziehung zwischen Fehler und Kosten. Desweiteren bestimmen wir das asymptotische Verhalten der entsprechenden minimalen Fehler. Wir zeigen die Abhängigkeit der minimalen Fehler von der räumlichen Dimension d, vom Glättungseffekt der vom Driftterm erzeugten Halbgruppe, von der durch den Diffusionsterm festgelegten Kopplung des unendlichdimensionalen Systems skalarer Ornstein-Uhlenbeck-Prozesse und von dem Zerfall der Eigenwerte von W im Falle nuklearen Rauschens. Asymptotische Optimalität wird erreicht durch implizite Euler-Maruyama-Verfahren, versehen mit nicht-uniformen Zeitdiskretisierungen. Diese Optimalität kann nicht notwendigerweise durch uniforme Zeitdiskretisierungen erreicht werden, welche häufig in der Literatur verwendet werden. Wir ergänzen unsere theoretischen Resultate durch numerische Untersuchungen.

Contents

Chapter 1

Introduction

The topic of this work is the pointwise approximation in a strong sense of infinite-dimensional Ornstein-Uhlenbeck processes. Such processes X are of the form

$$X(t) = \sum_{j \in \mathbb{N}^d} Y_j(t) \cdot h_j, \quad t \in [0, \infty),$$

with $d \in \mathbb{N}$, where $(h_j)_{j \in \mathbb{N}^d}$ forms an orthonormal basis of a separable Hilbert space H and $(Y_j)_{j \in \mathbb{N}^d}$ is a family of scalar, generally coupled, Ornstein-Uhlenbeck processes. Moreover, those H-valued processes are mild solutions of particular stochastic evolution equations with additive noise of the form

$$dX(t) = AX(t)\,dt + B(t)\,dW(t)$$

in which the coefficients satisfy specific assumptions. This equation is a special case of the more general stochastic parabolic type equation with multiplicative noise

$$dX(t) = (AX(t) + f(t, X(t))\,dt + B(t, X(t))\,dW(t) \tag{1.1}$$

on H. Here A denotes the infinitesimal generator of a strongly continuous semigroup and $W = (W(t))_{t \geq 0}$ is a (cylindrical) Wiener process. The mappings f and B satisfy suitable assumptions such that a unique mild solution $X = (X(t))_{t \geq 0}$ of (1.1) exists and is given as an H-valued continuous stochastic process, namely an infinite-dimensional Ornstein-Uhlenbeck process in these studies.

Historically, the first methods for numerical approximation of parabolic stochastic partial differential equations of type (1.1) are analyzed in [GK96] and [GN97]. These papers were followed by a lot of further contributions about this topic. For a detailed overview of the literature see, e.g., [JK09b]. Here we state as a partial list of contributions concerning the calculation of upper error bounds of specific algorithms the works [ANZ98], [S99], [DG01], [KS01], [H02], [H03], [MGR07b], [MGRW07] and [MGRW08]. The approximation schemes used in those articles are based on a finite number of one-dimensional components of the driving Wiener process W. Upper error bounds do not answer the question whether an algorithm is the best possible one out of a class of approximations for the solution. For the answer it is necessary to estimate a lower error bound. The first lower error bounds for equations of type (1.1) are derived in [DG01] followed by [MGR07a], [MGR07b], [MGRW07] and [MGRW08].

We approximate in this work the stochastic evolution equation of type (1.1) with additive noise

$$\begin{aligned} dX(t) &= AX(t)\,dt + B\,dW(t), \quad t \in [0,T], \\ X(0) &= \xi, \end{aligned} \tag{1.2}$$

on the Hilbert space $H = L_2\left((0,1)^d\right)$ with $d \in \mathbb{N}$. That means that $f = 0$ and that B does not depend on the process X. Moreover, W denotes a Q-Wiener process on H if its covariance Q is a trace class operator, or otherwise a cylindrical Wiener process on H. Furthermore, the initial value $\xi \in H$ is assumed to be deterministic for simplicity. The mild solution X of (1.2) is given as an infinite-dimensional Ornstein-Uhlenbeck process and we are interested in its approximation at a fixed single time point $T > 0$. For this reason we construct approximations $\widehat{X}_N(T)$ to $X(T)$ that use at most a total number of $N \in \mathbb{N}$ evaluations in time of a finite number of the one-dimensional components $\langle W, h_j \rangle$ of the driving Wiener process W. Here $(h_j)_{j\in\mathbb{N}^d}$ forms a complete orthonormal system in H, which also is a sequence of eigenfunctions of the operators Q and A. We consider N to be the cost of such an algorithm and our aim is to construct algorithms with an optimal relation between the error and the cost. As a criterion how close the approximation is to the solution, we measure for every realization the distance between $X(T)$ and $\widehat{X}_N(T)$ in the L_2-norm and then average over all trajectories. Therefore, the error of an approximation $\widehat{X}_N(T)$ is defined by

$$\mathrm{e}\left(\widehat{X}_N(T)\right) = \left(\mathrm{E}\left\|X(T) - \widehat{X}_N(T)\right\|^2\right)^{1/2}.$$

Furthermore, we define the Nth minimal error

$$\mathrm{e}_N = \inf_{\widehat{X}_N(T)} \mathrm{e}\left(\widehat{X}_N(T)\right).$$

This is the smallest possible error of any such algorithm $\widehat{X}_N(T)$. For the approximation error, we establish lower and upper bounds in a weakly asymptotic sense as $N \to \infty$ without the corresponding asymptotic constants. Thus, to avoid in our assumptions and results positive constants that only depend on the equation we use the notation $f_n \preceq g_n$, which means $\sup_{n \in \mathcal{N}} f_n/g_n < \infty$ for sequences of positive reals f_n and g_n with respect to a countable index set $\mathcal{N}$. Moreover, $f_n \asymp g_n$ means $f_n \preceq g_n$ and $g_n \preceq f_n$. Now, we state further conditions on A, B and W in (1.2) we assume in these notes.

Let Q be the covariance operator of W satisfying

$$Qh_j = \lambda_j \cdot h_j$$

with

$$\lambda_j \asymp |j|_2^{-\gamma}$$

for every $j \in \mathbb{N}^d$ and a fixed parameter $\gamma \in \{0\} \cup (d, \infty)$. In the case that $\gamma > d$ we call (1.2) an equation with nuclear (or trace class) noise whereas if $\gamma = 0$ we call (1.2) an equation with space-time white noise and assume further $d = 1$ to guarantee existence of the mild solution. In the sequel, these two cases are shortly denoted by (TC) and (ID). Note that larger values of γ lead to higher smoothness of the noise and the solution.

Let $A : D(A) \subset H \to H$ be a linear operator, satisfying

$$Ah_j = -\mu_j \cdot h_j$$

with

$$\mu_j \asymp |j|_2^{\alpha}$$

for every $j \in \mathbb{N}^d$ and a fixed parameter $\alpha \geq 2$, as well as

$$D(A) = \left\{ h \in H \,\middle|\, \sum_{j \in \mathbb{N}^d} |\mu_j|^2 \cdot |\langle h, h_j \rangle|^2 < \infty \right\}.$$

Let B be an operator, satisfying

$$1 \preceq \langle Bh_i, h_i \rangle^2 \tag{1.3}$$

and

$$\langle Bh_i, h_j \rangle^2 \preceq \begin{cases} \prod_{\substack{\ell=1 \\ i_\ell \neq j_\ell}}^{d} |i_\ell - j_\ell|^{-\beta}, & \text{if } i \neq j, \\ 1, & \text{if } i = j, \end{cases} \tag{1.4}$$

for every $i, j \in \mathbb{N}^d$ and a fixed parameter $\beta > 1$. Note that larger values of β lead to a faster decay of the scalar product away from the diagonal elements.

Due to our assumptions, the mild solution X of equation (1.2) at the time T is given by

$$X(T) = \sum_{j \in \mathbb{N}^d} Y_j(T) \cdot h_j,$$

where the real-valued processes Y_j, with $j \in \mathbb{N}^d$, are coupled Ornstein-Uhlenbeck processes, satisfying

$$\begin{aligned} dY_j(t) &= -\mu_j \cdot Y_j(t)\, dt + \sum_{i \in \mathbb{N}^d} |i|_2^{-\gamma/2} \cdot \langle Bh_i, h_j \rangle \, d\beta_i(t), \quad t \in [0, T], \\ Y_j(0) &= \langle \xi, h_j \rangle . \end{aligned}$$

Our assumptions are weaker than the ones given in [MGRW08] where the authors consider a stochastic heat equation with the identity operator as diffusion and a special choice of the orthonormal basis of H. For instance, by our requirements, pointwise multiplication operators of the form $Bh = g \cdot h$ are allowed as diffusion for $h \in H$ with a sufficiently smooth function $g : [0, 1]^d \to \mathbb{R}$. The assumption that the operator A in the drift term and the covariance operator Q use the same system of eigenfunctions is also assumed in, e.g., [H03], [LR04], [Y04], [MGR07a] and [MGR07b].

The analysis of minimal errors in [MGRW08] prove in particular that weakly asymptotic optimality cannot be achieved by algorithms using an uniform time discretization, which is a very common approach in literature. These algorithms use for a finite index set $\mathcal{I} \subset \mathbb{N}^d$ evaluations of every component $\langle W, h_i \rangle$ with $i \in \mathcal{I}$ at the time nodes

$$t_k = \frac{k}{n} \cdot T, \quad k = 1, \ldots, n. \tag{1.5}$$

The authors show that it is crucial to consider a non-uniform time discretization or even a non-equidistant time discretization to gain optimality. They do so by introducing different classes of algorithms, which use different time discretizations. Then, they give sharp lower and upper error bounds for the minimal errors in every algorithm class. Moreover, they provide algorithms $\widehat{X}_N(T)$, which are weakly asymptotically optimal, i.e. $\mathrm{e}(\widehat{X}_N(T)) \asymp \mathrm{e}_N$, in the respective classes.

In this work, we follow this approach by defining four different classes of algorithms consisting of approximations $\widehat{X}_N(T)$ that use different time discretizations. Let $\mathfrak{X}_N^{\mathrm{uni}}$ denote the class of algorithms with uniform time discretization where its elements use the time nodes (1.5). We enlarge this class by defining on the one hand the class $\mathfrak{X}_N^{\mathrm{equi}}$ of algorithms with equidistant time discretization whose elements use the time nodes

$$t_{k,i} = \frac{k}{n_i} \cdot T, \quad k = 1, \ldots, n_i,$$

for every $i \in \mathcal{I}$ with a variable number n_i for the evaluation of $\langle W, h_i \rangle$. On the the other hand we define the class $\mathfrak{X}_N^{\#}$ of algorithms where the nodes

$$0 < t_{1,i} < \ldots < t_{n,i} \leq T$$

can be freely chosen with a fixed number of n for the evaluation of every $\langle W, h_i \rangle$ with $i \in \mathcal{I}$. As the largest class we define the class $\mathfrak{X}_N^{*}$ of algorithms, which allows its elements to use any choice of the nodes

$$0 < t_{1,i} < \ldots < t_{n_i,i} \leq T$$

with the variable number of n_i for the evaluation of $\langle W, h_i \rangle$ for every $i \in \mathcal{I}$. For the corresponding Nth minimal error, we consider

$$\mathrm{e}_N^{\diamond} = \inf \left\{ \mathrm{e}\left(\widehat{X}_N(T)\right) \,\middle|\, \widehat{X}_N(T) \in \mathfrak{X}_N^{\diamond} \right\}$$

where $\diamond \in \{*, \#, \mathrm{equi}, \mathrm{uni}\}$. We study the weakly asymptotic behaviour of the minimal errors with respect to the cost N and provide weakly asymptotically optimal approximation schemes in the classes of algorithms depending on the parameters d, α, β and γ.

The first main result covers the case $B = I$, where I is the identity operator on H, i.e. the limiting case $\beta \to \infty$. This leads to independent real-valued Ornstein-Uhlenbeck processes as coefficients in the Fourier series of X and extends the results of [MGRW08]. Here we obtain

$$\mathrm{e}_N^{\diamond} \asymp N^{-P_\diamond}$$

with

$$P_* = \begin{cases} (\gamma+\alpha-d)/(2d), & \text{if } \gamma+\alpha < 3d, \\ 1, & \text{if } \gamma+\alpha > 3d, \end{cases}$$

$$P_\# = (\gamma+\alpha-d)/(\gamma+\alpha+d),$$

$$P_{\mathrm{equi}} = \begin{cases} (\gamma+\alpha-d)/(2(\alpha+d)), & \text{if } \gamma-\alpha < 3d, \\ 1, & \text{if } \gamma-\alpha > 3d, \end{cases}$$

$$P_{\mathrm{uni}} = \begin{cases} (\gamma+\alpha-d)/(2(\alpha+d)), & \text{if } \gamma-\alpha < d, \\ (\gamma+\alpha-d)/(\gamma+\alpha+d), & \text{if } \gamma-\alpha > d. \end{cases}$$

For the limiting cases, which are not covered above, we also provide asymptotically optimal Nth minimal errors containing logarithmic factors. Furthermore, we introduce asymptotically optimal algorithms $\widehat{X}_N^{\diamond}(T) \in \mathfrak{X}_N^{\diamond}$ that achieve $\mathrm{e}(\widehat{X}_N^{\diamond}(T)) \asymp \mathrm{e}_N^{\diamond}$ for every $\diamond \in \{*, \#, \mathrm{equi}, \mathrm{uni}\}$. We conclude in the (ID) case and in the (TC) case with smaller smoothness that the constructed approximation schemes using a non-equidistant time discretization are superior over all those algorithms using equidistant time nodes. Further, we see that in case of nuclear noise with higher smoothness, the classes $\mathfrak{X}_N^{\mathrm{uni}}$ and $\mathfrak{X}_N^{\#}$ are of the same quality and suboptimal with respect to the classes $\mathfrak{X}_N^{\mathrm{equi}}$ and $\mathfrak{X}_N^{*}$.

For the second main result we return to the more general operators B satisfying the conditions (1.3) and (1.4). At first, we consider the case $d = 1$ and show

$$\mathrm{e}_N^{\diamond} \asymp N^{-P_\diamond}$$

for $\diamond \in \{\#, \mathrm{uni}\}$ with

$$P_\# = (\gamma+\alpha-1)/(\gamma+\alpha+1),$$

if

$$\gamma + \alpha > 3 \ \text{ and } \ \max(\alpha, \gamma) \leq \beta$$

or

$$\beta + \alpha > 3 \ \text{ and } \ \alpha \leq \beta \leq \gamma,$$

as well as

$$P_{\text{uni}} = \begin{cases} (\gamma + \alpha - 1)/(2(\alpha + 1)), & \text{if } \ \gamma - \alpha < 1 \ \text{ and } \ \max(\alpha, \gamma) \leq \beta, \\ (\gamma + \alpha - 1)/(\gamma + \alpha + 1), & \text{if } \ \min(\beta, \gamma) - \alpha > 1. \end{cases}$$

The corresponding optimal algorithms $\widehat{X}_N^{\diamond}(T) \in \mathfrak{X}_N^{\diamond}$ achieving $\mathrm{e}(\widehat{X}_N^{\diamond}(T)) \asymp \mathrm{e}_N^{\diamond}$ with $\diamond \in \{\#, \text{uni}\}$ are presented in the case that the parameters α, β and γ satisfy the respective stated conditions. For further combinations of those parameters we provide algorithms in the class $\mathfrak{X}_N^{\#}$, which are not proven to be optimal, but superior over all algorithms with uniform time discretization and we give an overview of these parameters. Additionally, for the remaining parameters we construct algorithms in both of the classes, which are not proven to be optimal and yield their upper error bounds. As in the first result for $B = I$, we see that in the (ID) case as well as in the (TC) case with smaller values of γ all the approximation schemes with uniform time discretization are inferior to $\widehat{X}_N^{\#}(T)$.

In the third main result we study the case $d \in \mathbb{N} \setminus \{1\}$ and obtain

$$\mathrm{e}_N^{\diamond} \preceq N^{-P_{\diamond}} \cdot (\ln N)^{(d-1)/2}$$

for $\diamond \in \{\#, \text{uni}\}$ with

$$P_{\#} = (\gamma + \alpha - d)/(\gamma + \alpha + d),$$

if

$$\gamma + \alpha > 3d \ \text{ and } \ \max(\alpha, \gamma) \leq \beta$$

or

$$\beta + \alpha > 3d \ \text{ and } \ \alpha \leq \beta \leq \gamma,$$

as well as

$$P_{\text{uni}} = (\gamma + \alpha - d)/(\gamma + \alpha + d),$$

if

$$\alpha \leq d, \ \ \gamma \geq \beta \cdot d \ \text{ and } \ \beta - \alpha > d,$$

where the given upper bounds are weakly asymptotically optimal up to the logarithmic factor. As in the case $d = 1$, we provide the corresponding algorithms. Also, we construct superior algorithms in the class $\mathfrak{X}_N^{\#}$ for further combinations of the parameters d, α, β and γ and give an overview of those. In addition, we construct algorithms, which are not proven to be optimal for remaining parameters and give their upper error bounds. Here we see that for large smoothness the algorithms in the classes $\mathfrak{X}_N^{\#}$ and $\mathfrak{X}_N^{\mathrm{uni}}$ are of the same quality while for small values of γ and large β the class $\mathfrak{X}_N^{\mathrm{uni}}$ is suboptimal with respect to $\widehat{X}_N^{\#}(T)$.

Furthermore, we show that the upper error bounds, which are stated in the second and third main result, also hold for time dependent diffusion operators $B(t)$, $t \in [0, T]$, satisfying

$$\sup_{t\in[0,T]} \langle B(t)h_i, h_j\rangle^2 \preceq \begin{cases} \prod\limits_{\substack{\ell=1 \\ i_\ell \neq j_\ell}}^{d} |i_\ell - j_\ell|^{-\beta}, & \text{if } i \neq j, \\ 1, & \text{if } i = j, \end{cases}$$

for every $i, j \in \mathbb{N}^d$ and a fixed parameter $\beta > 1$.

The established algorithms with non-equidistant time discretizations are based on the drift-implicit Euler-Maruyama scheme using time nodes given by the quantiles with respect to a fixed density, the so-called regular time discretization. In comparison to the complete characterization of the asymptotically optimal order of convergence for the approximation of the stochastic evolution equation (1.2) in the case $B = I$, we only present partial results in case of a more general diffusion. It remains to determine sharp error bound of the minimal error for several values of the parameters d, α, β and γ in the classes $\mathfrak{X}_N^{\mathrm{uni}}$ and $\mathfrak{X}_N^{\#}$ as well as the research of the classes $\mathfrak{X}_N^{\mathrm{equi}}$ and $\mathfrak{X}_N^{*}$.

The results in [MGRW07], [MGRW08] and in this work about weakly asymptotically optimal algorithms for pointwise approximation differ from those that use a global approximation error criterion. In [MGR07a] the authors study algorithms $\widehat{X}_N$ for the mild solution of (1.2) with respect to the error

$$\mathrm{e}\left(\widehat{X}_N\right) = \left(\mathrm{E}\int_0^T \left\|X(t) - \widehat{X}_N(t)\right\|^2 dt\right)^{1/2}$$

and calculate the Nth minimal errors. Here it is sufficient to consider approximation schemes with equidistant time discretization to obtain weakly asymptotic optimality.

The analysis of minimal errors is a main topic for continuous problems, i.e. in information-based complexity theory. See, e.g., [N88], [TWW88] and [R00] for results and further references. Results about the minimal errors of finite dimensional stochastic differential equations are given in, e.g., [HMGR01], [MG02a], [MG02b], [MG04], [N06] and [MGR08]. In the latter article also results are given about the weak approximation of the solution X, i.e. the approximation of functionals of the form $t \to \mathrm{E}(h(X(t)))$ for a suitable real-valued mapping h.

These notes are organized as follows. In Chapter 2 we give a short overview of definitions and facts on stochastic partial differential equations of evolutionary type. Furthermore, examples for operators in the considered stochastic evolution equation are given as well as a small survey about several known approximation results in the literature. In Chapter 3 we introduce the classes of approximations, which we analyze and the concept of minimal errors. Thereafter, we construct algorithms in the different classes and state the main results about their optimality. In addition, we state error bounds for the minimal error. At the end of this chapter, we give the proofs of the results. In Chapter 4 we complement our theoretical results by the simulation of trajectories and providing computational average errors for some of the stated approximations. In the Appendices A and B we recall some basic facts from functional analysis about linear operators and in Appendix C we state some auxiliary results we use in our proofs.

Chapter 2

Stochastic Evolution Equations

This chapter provides a short summary of the theory of stochastic partial differential equations of evolutionary type based on the semigroup approach. The definitions and conclusions are mainly taken from [DPZ92] and, concerning Wiener processes and stochastic integration, from [PR07]. The Bochner integral is introduced in, e.g., Appendix E in [C80], Appendix C in [EN00] or Appendix A in [PR07]. The definitions and results concerning the theory of linear operators are summarized in the Appendices A and B.

We use the following notation throughout the rest of the chapter. For a topological vector space V its Borel σ-algebra is denoted by $\mathcal{B}(V)$. For a probability space $(\Omega, \mathcal{F}, P)$ we set

$$\mathrm{E}(Y) = \int_\Omega Y(\omega)\, P(d\omega)$$

for an $\mathcal{F}$-measurable function $Y : \Omega \to \mathbb{R}$ provided that $\int_\Omega |Y(\omega)|\, P(d\omega) < \infty$. Moreover, let $(U, \|\cdot\|_U, \langle\cdot,\cdot\rangle_U)$ and $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ be two separable real Hilbert spaces as well as $\mathcal{L}(U,H)$ and $\mathcal{L}_{\mathrm{nuc}}(U,H)$ denotes respectively the class of bounded linear operators and the class of nuclear operators mapping U to H.

2.1 Wiener Processes on Hilbert Spaces

Definition 2.1.1 ***(Gaussian measure)***
A probability measure μ on $(U, \mathcal{B}(U))$ is called Gaussian measure if its characteristic function $\widehat{\mu}$ satisfies

$$\widehat{\mu}(u) = \int_U \exp\left(i \cdot \langle u, v\rangle_U\right) \, \mu(dv) = \exp\left(i \cdot \langle m, u\rangle_U - \frac{1}{2} \cdot \langle Qu, u\rangle_U\right)$$

for every $u \in U$, where $i = \sqrt{-1}$ and:

- *$m \in U$ is called mean of μ.*
- *$Q \in \mathcal{L}_{\mathrm{nuc}}(U) = \mathcal{L}_{\mathrm{nuc}}(U, U)$ is non-negative and symmetric (hence a trace class operator), and called covariance operator of μ.*

A Gaussian measure μ is uniquely determined by m and Q and also be denoted by $N(m, Q)$. The reason for calling m the mean and Q the covariance of μ is provided by the properties

$$\int_U \langle x, h\rangle_U \, \mu(dx) = \langle m, h\rangle_U$$

and

$$\int_U \left(\langle x, h\rangle_U - \langle m, h\rangle_U\right)\left(\langle x, g\rangle_U - \langle m, g\rangle_U\right) \mu(dx) = \langle Qh, g\rangle_U$$

for every $h, g \in U$. Furthermore, it holds for every $h \in U$

$$\langle Qh, h\rangle_U = \int_U \langle x, h\rangle^2 \, \mu(dx) - \left(\int_U \langle x, h\rangle \, \mu(dx)\right)^2,$$

$$\left(\int_U \langle x, h\rangle \, \mu(dx)\right)^2 \leq \int_U \langle x, h\rangle^2 \, \mu(dx)$$

and

$$\int_U \|x - m\|_U^2 \, \mu(dx) = \mathrm{tr}(Q).$$

For the existence of a Gaussian measure we get the following result.

Proposition 2.1.1 *Let $Q \in \mathcal{L}(U) = \mathcal{L}(U, U)$ be a trace class operator and $m \in U$. Then there exists a Gaussian measure $\mu = N(m, Q)$ on $(U, \mathcal{B}(U))$.*

Proof: See, e.g., Corollary 2.1.7. in [PR07]. □

Definition 2.1.2 ***(Gaussian random variable)***
Let $Q \in \mathcal{L}(U)$ be a trace class operator and $m \in U$. A U-valued random variable X on $(\Omega, \mathcal{F}, P)$ is called Gaussian with mean m and covariance Q, if $P \circ X^{-1} = N(m, Q)$.

For a Gaussian random variable X with mean m and covariance Q, $\langle X, u\rangle_U$ is normally distributed for every $u \in U$, and the following properties hold.

- $\mathrm{E}\left(\langle X, u\rangle_U\right) = \langle m, u\rangle_U$ for every $u \in U$.
- $\mathrm{E}\left(\langle X - m, u\rangle_U \cdot \langle X - m, v\rangle_U\right) = \langle Qu, v\rangle_U$ for every $u, v \in U$.
- $\mathrm{E}\left(\|X - m\|_U^2\right) = \mathrm{tr}(Q)$.

For the representation of such a Gaussian random variable, we get the following result.

Proposition 2.1.2 *Let $Q \in \mathcal{L}(U)$ be a trace class operator, $m \in U$ and $(e_i)_{i\in\mathfrak{I}}$ be an orthonormal basis of U consisting of eigenvectors of Q with corresponding non-negative eigenvalues $(\lambda_i)_{i\in\mathfrak{I}}$. Then for a U-valued random variable X on $(\Omega, \mathcal{F}, P)$ the following assertions are equivalent.*

i) *X is a Gaussian random variable with mean m and covariance Q.*

ii)
$$X = \sum_{i\in\mathfrak{I}} \sqrt{\lambda_i} \cdot \beta_i \cdot e_i + m, \tag{2.1}$$
where $(\beta_i)_{i\in\mathfrak{I}}$ is an independent family of real-valued $N(0,1)$-distributed random variables, i.e. $P \circ \beta_i^{-1} = N(0, 1)$ for every $i \in \mathfrak{I}$.

In both cases, the series (2.1) converges in $L_2(\Omega, \mathcal{F}, P; U)$.

Proof: See, e.g., Proposition 2.1.6. in [PR07]. □

Definition 2.1.3 ***(Q-Wiener process)***
Let $T > 0$ and $Q \in \mathcal{L}(U)$ be a trace class operator. A U-valued stochastic process $(W(t))_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, P)$ is called a Q-Wiener process if the following properties hold.

- *$W(0) = 0$.*
- *W has P-a.s. continuous trajectories, i.e. $t \mapsto W(t)$ is continuous P-a.s.*
- *The increments of W are independent, i.e. for every $0 = t_0 \leq t_1 < \ldots < t_n \leq T$ with $n \in \mathbb{N}$, the random variables*

$$W(t_i) - W(t_{i-1}), \quad i = 1, \ldots, n,$$

  are independent.
- *The increments of W are $N(0, (t-s)Q)$-distributed, i.e. they have the Gaussian laws*

$$P \circ (W(t) - W(s))^{-1} = N(0, (t-s)Q)$$

  for every $0 \leq s < t \leq T$.

For the representation of a Q-Wiener process, we get the following result.

Proposition 2.1.3 *Let $T > 0$, $Q \in \mathcal{L}(U)$ be a trace class operator and $(e_i)_{i \in \mathfrak{I}}$ be an orthonormal basis of U consisting of eigenvectors of Q with corresponding non-negative eigenvalues $(\lambda_i)_{i \in \mathfrak{I}}$. Then a Q-Wiener process exists and the following assertions are equivalent.*

i) $(W(t))_{t \in [0,T]}$ *is a Q-Wiener process on $(\Omega, \mathcal{F}, P)$.*

ii)

$$W(t) = \sum_{i \in \mathfrak{I}} \sqrt{\lambda_i} \cdot \beta_i(t) \cdot e_i, \tag{2.2}$$

where $(\beta_i)_{i \in \mathfrak{I}}$ is an independent family of standard one-dimensional Brownian motions on $(\Omega, \mathcal{F}, P)$.

In both cases, the series converges in $L_2(\Omega, \mathcal{F}, P; C([0,T], U))$.

Proof: See, e.g., Proposition 2.1.10. in [PR07]. □

An increasing family $(\mathcal{F}_t)_{t \geq 0}$ of σ-algebras is called a filtration on a probability space $(\Omega, \mathcal{F}, P)$ if $\mathcal{F}_t \subset \mathcal{F}$ for every $t \geq 0$. The σ-algebra $\mathcal{F}_t$ can be interpreted as the information at the time t. Now, further demands on a filtration are needed.

Definition 2.1.4 ***(Normal filtration)***
A filtration $(\mathcal{F}_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$ is called a normal filtration if the following properties hold.

- *$\mathcal{F}_0$ contains every P-null set, i.e. if $A \in \mathcal{F}$ and $P(A) = 0$, then $A \in \mathcal{F}_0$.*
- *$(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, i.e.*

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s \quad \text{for every } t \geq 0.$$

Definition 2.1.5 ***(Q-Wiener process with respect to a filtration)***
A Q-Wiener process $(W(t))_{t\in[0,T]}$ is called a Q-Wiener process with respect to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ if the following properties hold.

- *The process $(W(t))_{t\in[0,T]}$ is adapted to $(\mathcal{F}_t)_{t\in[0,T]}$, i.e. $W(t)$ is $\mathcal{F}_t$-measurable for every $t \in [0, T]$.*
- *The increment $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for every $0 \leq s < t \leq T$.*

Proposition 2.1.4 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{N} = \{A \in \mathcal{F} \mid P(A) = 0\}$ be the set of P-null sets, $\mathcal{F}_t^0 = \sigma(W(s) \mid s \in [0, t])$ be the σ-algebra generated by the Q-Wiener process $(W(t))_{t\in[0,T]}$ and $\widetilde{\mathcal{F}}_t^0 = \sigma(\mathcal{F}_t^0 \cup \mathcal{N})$. Then $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$ with $\widetilde{\mathcal{F}}_t = \bigcap_{s>t} \widetilde{\mathcal{F}}_s^0$ is a normal filtration and $(W(t))_{t\in[0,T]}$ is a Q-Wiener process with respect to the normal filtration $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$.*

Proof: See, e.g., Proposition 2.1.13. in [PR07]. □

As a preliminary for the introduction of stochastic integration in Hilbert spaces, we define martingales with values in a separable real Banach space B similar as in the real-valued case.

Definition 2.1.6 ***(Conditional expectation)***
Let B be a separable real Banach space, $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra and $X : \Omega \to B$ be an $\mathcal{F}$-measurable and Bochner integrable mapping. Then a $\mathcal{G}$-measurable mapping $Z : \Omega \to B$ satisfying

$$\int_A Z \, dP = \int_A X \, dP$$

for every $A \in \mathcal{G}$ is denoted by $\mathrm{E}(X \,|\, \mathcal{G})$ and called the conditional expectation of X given $\mathcal{G}$.

The justification for this definition is given by the following result.

Proposition 2.1.5 *Let B be a separable real Banach space, $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra and $X : \Omega \to B$ be an $\mathcal{F}$-measurable and Bochner integrable mapping. Then there exists a unique, up to a set of P-probability zero, conditional expectation of X given $\mathcal{G}$. Furthermore, it holds*

$$\|\mathrm{E}(X \,|\, \mathcal{G})\|_B \leq \mathrm{E}(\|X\|_B \,|\, \mathcal{G}).$$

Proof: See, e.g., Proposition 2.2.1. in [PR07]. □

Definition 2.1.7 ***(Martingale)***
Let B be a separable real Banach space, $(\mathcal{F}_t)_{t \geq 0}$ be a filtration on a probability space $(\Omega, \mathcal{F}, P)$ and $(M(t))_{t \geq 0}$ be a B-valued stochastic process on $(\Omega, \mathcal{F}, P)$. The process $(M(t))_{t \geq 0}$ is called an $(\mathcal{F}_t)_{t \geq 0}$-martingale if the following properties hold.

- $\mathrm{E}(\|M(t)\|_B) < \infty$ *for every* $t \geq 0$.
- $M(t)$ *is* $\mathcal{F}_t$*-measurable for every* $t \geq 0$.
- $\mathrm{E}(M(t) \,|\, \mathcal{F}_s) = M(s)$ *for every* $0 \leq s \leq t < \infty$.

For a fixed $T > 0$ we denote the space of all B-valued continuous, square integrable $(\mathcal{F}_t)_{t \in [0,T]}$-martingales $(M(t))_{t \in [0,T]}$ by $\mathcal{M}_T^2(B)$ or $\mathcal{M}_T^2$. By Proposition 2.2.9. in [PR07] it follows that the space $\mathcal{M}_T^2$ equipped with the norm

$$\|M\|_{\mathcal{M}_T^2} = \sup_{t \in [0,T]} \left(\mathrm{E}(\|M(t)\|_B^2)\right)^{1/2} = \left(\mathrm{E}(\|M(T)\|_B^2)\right)^{1/2}$$

is a Banach space and the martingale inequality

$$\|M\|_{\mathcal{M}_T^2} \leq \left(\mathrm{E}\left(\sup_{t \in [0,T]} \|M(t)\|_B^2\right)\right)^{1/2} \leq 2 \cdot \left(\mathrm{E}(\|M(T)\|_B^2)\right)^{1/2}.$$

For the martingale property of a Q-Wiener process, we get the following result.

Proposition 2.1.6 *Let $T > 0$ and $(W(t))_{t\in[0,T]}$ be a U-valued Q-Wiener process with respect to a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, P)$. Then $(W(t))_{t\in[0,T]}$ is a U-valued continuous, square integrable $(\mathcal{F}_t)_{t\in[0,T]}$-martingale, i.e. $W \in \mathcal{M}_T^2(U)$, with $\mathrm{E}(\|W(t)\|_U^2) = t \cdot \mathrm{tr}(Q) < \infty$ for every $t \in [0,T]$.*

Proof: See, e.g., Proposition 2.2.10. in [PR07]. □

2.2 Stochastic Integration

In this section we define the stochastic integral $\int \Phi(t)\, dW(t)$. The construction differs from the classical vector-valued integrals, because the trajectories $t \mapsto W(t)$ are not differentiable and not of bounded variation. We follow the one in Section 2.3. in [PR07] using four steps. Therefore, we fix $T > 0$, a probability space $(\Omega, \mathcal{F}, P)$ and a Q-Wiener process $(W(t))_{t\in[0,T]}$ with respect to a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

Step 1: Integration of elementary processes

$$\Phi(t) = \sum_{m=0}^{k-1} \Phi_m \cdot 1_{(t_m, t_{m+1}]}(t) \tag{2.3}$$

where:

- $k \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_k = T$.
- $\Phi_m : \Omega \to \mathcal{L}(U, H)$ is $\mathcal{F}_{t_m}$-measurable and bounded.

Let $\mathcal{E}$ be the set of all elementary processes of type (2.3) and define

$$\int_0^t \Phi(s)\, dW(s) = \sum_{m=0}^{k-1} \Phi_m \left(W(t_{m+1} \wedge t) - W(t_m \wedge t)\right), \quad t \in [0,T]. \tag{2.4}$$

This induces a linear mapping

$$\begin{aligned} \mathrm{Int} : \mathcal{E} &\to \mathcal{M}_T^2(H), \\ \Phi &\mapsto \int_0^t \Phi(s)\, dW(s), \quad t \in [0,T]. \end{aligned}$$

Thus, the stochastic integral $\int_0^t \Phi(s)\, dW(s)$, $t \in [0,T]$, is an H-valued continuous, square integrable $(\mathcal{F}_t)_{t\in[0,T]}$-martingale.

Step 2: The Ito isometry

$$\mathrm{E}\left(\left\|\int_0^t \Phi(s)\, dW(s)\right\|_H^2\right) = \mathrm{E}\left(\int_0^t \left\|\Phi(s)Q^{1/2}\right\|_{\mathcal{L}_{\mathrm{HS}}(U,H)}^2 ds\right), \quad t \in [0,T], \tag{2.5}$$

holds for every $\Phi \in \mathcal{E}$, where $\|\cdot\|_{\mathcal{L}_{\mathrm{HS}}(U,H)}$ denotes the Hilbert-Schmidt norm on the space $\mathcal{L}_{\mathrm{HS}}(U,H)$ of all Hilbert-Schmidt operators from U to H. Recall from Appendix A that $(\mathcal{L}_{\mathrm{HS}}(U,H), \|\cdot\|_{\mathcal{L}_{\mathrm{HS}}(U,H)}, \langle\cdot,\cdot\rangle_{\mathcal{L}_{\mathrm{HS}}(U,H)})$ is a separable Hilbert space. Now, we rewrite the terms in equation (2.5). To this end, we define the separable Hilbert space $U_0 = Q^{1/2}(U)$ equipped with the scalar product

$$\langle u_0, v_0\rangle_{U_0} = \left\langle Q^{-1/2}u_0, Q^{-1/2}v_0\right\rangle_U,$$

where $Q^{-1/2}$ denotes the pseudo inverse of $Q^{1/2}$ if Q is not one-to-one. For more details, see, e.g., Appendix C in [PR07]. Note from Proposition A.0.10 in Appendix A that $Q^{1/2}$ is a Hilbert-Schmidt operator. Let $\mathcal{L}_{\mathrm{HS}}^0 = \mathcal{L}_{\mathrm{HS}}(U_0,H)$ be the separable Hilbert space of all Hilbert-Schmidt operators from U_0 to H. Thus,

$$\|A\|_{\mathcal{L}_{\mathrm{HS}}^0} = \|A \circ Q^{1/2}\|_{\mathcal{L}_{\mathrm{HS}}(U,H)}$$

for every $A \in \mathcal{L}_{\mathrm{HS}}^0$, implying $A|_{U_0} \in \mathcal{L}_{\mathrm{HS}}^0$ if $A \in \mathcal{L}_{\mathrm{HS}}(U,H)$. Then the Ito isometry (2.5) can be written in the form

$$\left\|\int_0^\cdot \Phi(s)\, dW(s)\right\|_{\mathcal{M}_T^2}^2 = \mathrm{E}\left(\int_0^T \|\Phi(s)\|_{\mathcal{L}_{\mathrm{HS}}^0}^2 ds\right) = \|\Phi\|_T^2,$$

where $\|\cdot\|_T$ is a seminorm on $\mathcal{E}$. Hence,

$$\mathrm{Int} : (\mathcal{E}, \|\cdot\|_T) \to (\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2})$$

is an isometric transformation and it follows that the definition of the stochastic integral can be extended to integrands contained in the abstract completion $\overline{\mathcal{E}}$ of $\mathcal{E}$ with respect to $\|\cdot\|_T$.

Step 3: An explicit representation of $\overline{\mathcal{E}}$ is given with the help of the product space

$\Omega_T = [0,T] \times \Omega$, the product $P_T = dt \otimes P$ of measures with the Lebesgue measure dt on $[0,T]$ and the predictable σ-algebra $\mathcal{P}_T$ on Ω_T defined by

$$\mathcal{P}_T = \sigma\left(\{(s,t] \times F_s \,|\, 0 \leq s < t \leq T, F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 \,|\, F_0 \in \mathcal{F}_0\}\right).$$

Note that a $\mathcal{P}_T$-measurable stochastic process is called predictable. Then

$$\begin{aligned} \overline{\mathcal{E}} &= \{\Phi : [0,T] \times \Omega \to \mathcal{L}_{\mathrm{HS}}^0 \,|\, \Phi \text{ is predictable and } \|\Phi\|_T < \infty\} \\ &= \mathcal{L}_2(\Omega_T, \mathcal{P}_T, P_T; \mathcal{L}_{\mathrm{HS}}^0) \end{aligned}$$

and $\mathrm{Int} : \mathcal{E} \to \mathcal{M}_T^2(H)$ can be uniquely extended to an isometry $\mathrm{Int} : \overline{\mathcal{E}} \to \mathcal{M}_T^2(H)$.

Step 4: A localization extends the definition of the stochastic integral to the linear space

$$\mathcal{N}_W = \left\{ \Phi : \Omega_T \to \mathcal{L}_{\mathrm{HS}}^0 \,\middle|\, \Phi \text{ is predictable and } P\left(\int_0^T \|\Phi(s)\|_{\mathcal{L}_{\mathrm{HS}}^0}^2 \, ds < \infty\right) = 1 \right\}$$

using suitable stopping times. $\mathcal{N}_W$ is called the class of stochastically integrable processes on $[0,T]$.

The construction of stochastic integrals $\int \Phi(t)\, dW(t)$ can be extended to the case that the covariance operator Q is not necessarily of finite trace. To this end, we extend the notion of a Q-Wiener process by the concept of cylindrical Wiener processes. In this thesis, we restrict our studies to the special case $Q = I$, where I is the identity operator on U. For this particular covariance, the representation (2.2) of a Q-Wiener process is of the form

$$W(t) = \sum_{i \in \mathfrak{I}} \beta_i(t) \cdot e_i$$

and this series does not converge in U for countable, infinite sets $\mathfrak{I}$. Nevertheless, with the help of a Hilbert-Schmidt operator $J : U \to U_1$ with respect to a Hilbert space $(U_1, \|\cdot\|_{U_1}, \langle \cdot, \cdot \rangle_{U_1})$, it is possible to define a Wiener process in U_1. First, due to the following result we mention that such a Hilbert space with a suitable Hilbert-Schmidt operator always exists, e.g. by the choice $U_1 = U$.

Proposition 2.2.1 *Let $(e_i)_{i\in\mathfrak{I}}$ be an orthonormal basis of U and $(a_i)_{i\in\mathfrak{I}} \in (0,\infty)^{\mathfrak{I}}$ be a sequence with $\sum_{i\in\mathfrak{I}} a_i^2 < \infty$. Define $U_1 = U$ and*

$$\begin{aligned} J : U &\to U_1, \\ u &\mapsto \sum_{i\in\mathfrak{I}} a_i \cdot \langle u, e_i\rangle_U \cdot e_i. \end{aligned}$$

Then J is one-to-one and a Hilbert-Schmidt operator.

Proof: See, e.g., Remark 2.5.1. in [PR07]. □

Next, we construct a Wiener process as stated in the following result.

Proposition 2.2.2 *Let $(e_i)_{i\in\mathfrak{I}}$ be an orthonormal basis of U, $(\beta_i)_{i\in\mathfrak{I}}$ be an independent family of standard one-dimensional Brownian motions and $J : U \to U_1$ be Hilbert-Schmidt, mapping into the Hilbert space $(U_1, \|\cdot\|_{U_1}, \langle\cdot,\cdot\rangle_{U_1})$. Then $Q_1 = JJ^* \in \mathcal{L}(U_1)$ is a trace class operator and the series*

$$W_1(t) = \sum_{i\in\mathfrak{I}} \beta_i(t) \cdot Je_i \tag{2.6}$$

converges in $\mathcal{M}_T^2(U_1)$ and defines a U_1-valued Q_1-Wiener process. Moreover, it holds

$$Q_1^{1/2}(U_1) = J(U) \tag{2.7}$$

and

$$\|u\|_U = \|Q_1^{-1/2} Ju\|_{U_1} = \|Ju\|_{Q_1^{1/2}(U_1)}$$

for every $u \in U$, i.e. $J : U \to Q_1^{1/2}(U_1)$ is an isometry.

Proof: See, e.g., Proposition 2.5.2. in [PR07]. □

The constructed Q_1-Wiener process (2.6) in U_1 is called a cylindrical Wiener process in U and depends on J. Now, we define the stochastic integral with respect to a cylindrical Wiener process, which basically is an integral with respect to the Q_1-Wiener process W_1 given by Proposition 2.2.2. Thus, we can integrate predictable $\mathcal{L}_{\mathrm{HS}}(Q_1^{1/2}(U_1), H)$-valued processes $\Phi = (\Phi(t))_{t\in[0,T]}$, which satisfy

$$P\left(\int_0^T \|\Phi(s)\|^2_{\mathcal{L}_{\mathrm{HS}}(Q_1^{1/2}(U_1),H)}\, ds < \infty\right) = 1.$$

However, we want to integrate processes with values in $\mathcal{L}_{\mathrm{HS}}(U,H)$. By Proposition 2.2.2, we have the equation (2.7) and that

$$\langle u,v\rangle_U = \left\langle Q_1^{-1/2}Ju, Q_1^{-1/2}Jv\right\rangle_{U_1} = \langle Ju, Jv\rangle_{Q_1^{1/2}(U_1)}$$

for every $u,v\in U$. Thus, $(Je_i)_{i\in\mathfrak{I}}$ is an orthonormal basis of $Q_1^{1/2}(U_1)$ and because of

$$\begin{aligned}\|\Phi\|^2_{\mathcal{L}_{\mathrm{HS}}(U,H)} &= \sum_{i\in\mathfrak{I}}\langle\Phi e_i,\Phi e_i\rangle_H\\ &= \sum_{i\in\mathfrak{I}}\left\langle\Phi\circ J^{-1}(Je_i),\Phi\circ J^{-1}(Je_i)\right\rangle_H = \|\Phi\circ J^{-1}\|^2_{\mathcal{L}_{\mathrm{HS}}(Q_1^{1/2}(U_1),H)},\end{aligned}$$

we conclude that

$$\Phi\in\mathcal{L}_{\mathrm{HS}}(U,H)\Longleftrightarrow\Phi\circ J^{-1}\in\mathcal{L}_{\mathrm{HS}}(Q_1^{1/2}(U_1),H),$$

i.e. that the stochastic integral $\int\Phi(t)\circ J^{-1}\,dW_1(t)$ with respect to the Q_1-Wiener process is well-defined. Now, we define the stochastic integral by

$$\int_0^t\Phi(s)\,dW(s) = \int_0^t\Phi(s)\circ J^{-1}\,dW_1(s),\quad t\in[0,T],\tag{2.8}$$

where the class of stochastically integrable processes on $[0,T]$ is given by

$$\mathcal{N}_W = \left\{\Phi:\Omega_T\to\mathcal{L}_{\mathrm{HS}}(U,H)\;\middle|\;\Phi\text{ is predictable and }P\left(\int_0^T\|\Phi(s)\|^2_{\mathcal{L}_{\mathrm{HS}}(U,H)}\,ds<\infty\right)=1\right\}.$$

Note that the stochastic integral defined by (2.8) does not depend on the choice of U_1 and J, because (2.8) is independent of J for elementary processes since (2.6).

The basic properties of the stochastic integral are stated, e.g., in Sections 4.4 to 4.7 in [DPZ92] and in Section 2.4. in [PR07]. In particular, it follows that the stochastic integral with respect to a U-valued Wiener process W with covariance Q can be represented in terms of one-dimensional stochastic integrals with respect to an independent family of standard one-dimensional Brownian motions $(\beta_i)_{i\in\mathfrak{I}}$ by

$$\int_0^T\Phi(t)\,dW(t) = \sum_{j\in\mathfrak{I}}\left(\sum_{i\in\mathfrak{I}}\lambda_i^{1/2}\cdot\int_0^T\langle\Phi(t)e_i,e_j\rangle_U\,d\beta_i(t)\right)\cdot e_j$$

for a stochastically integrable process $(\Phi(t))_{t\in[0,T]}$ with values in $\mathcal{L}_{\mathrm{HS}}(Q^{1/2}(U),U)$. In this expansion, $(e_i)_{i\in\mathcal{I}}$ denotes an orthonormal basis of U, $(\lambda_i)_{i\in\mathcal{I}}$ denotes a sequence of positive real numbers and it is required that $Qe_i = \lambda_i \cdot e_i$ for every $i \in \mathcal{I}$. See, e.g., Section 1.3 in [W08] for more details.

2.3 Existence and Uniqueness of Mild Solutions

In this section we introduce the concept of a mild solution for stochastic evolution equations of the type

$$\begin{aligned} dX(t) &= AX(t)\,dt + B(t,X(t))\,dW(t), \quad t\in[0,T], \\ X(0) &= \xi \in H, \end{aligned} \tag{2.9}$$

for a fixed $T > 0$. We distinguish between the two cases that W in (2.9) is either a Q-Wiener process or a cylindrical Wiener process with the identity as covariance. In the first case, we call (2.9) a stochastic partial differential equation with nuclear noise (or trace class noise), shortly denoted by (TC). In the second case, (2.9) is called a stochastic partial differential equation with space-time white noise and shortly denoted by (ID). In the (TC) case the further objects in (2.9) should fulfil the following conditions.

Assumption 2.3.1 ***(Assumptions in the (TC) case)***

- *The operator $A : D(A) \subset H \to H$ is the infinitesimal generator of the strongly continuous semigroup $(S(t))_{t\geq 0}$ on H.*
- *The operator $B : [0,T]\times H \to \mathcal{L}_{\mathrm{HS}}^0$ is measurable, where $\mathcal{L}_{\mathrm{HS}}^0 = \mathcal{L}_{\mathrm{HS}}(U_0,H)$ with $U_0 = Q^{1/2}(U)$.*
- *The operator B satisfies a Lipschitz condition and a linear growth condition, i.e. there exists a constant $c > 0$ such that*

$$\|B(t,h) - B(t,g)\|_{\mathcal{L}_{\mathrm{HS}}^0} \leq c \cdot \|h-g\|_H$$

and

$$\|B(t,h)\|_{\mathcal{L}_{\mathrm{HS}}^0} \leq c\cdot(1+\|h\|_H)$$

for every $t\in[0,T]$ and $h,g\in H$.

In the (ID) case, we assume the following conditions.

Assumption 2.3.2 ***(Assumptions in the (ID) case)***

- *The operator $A : D(A) \subset H \to H$ is the infinitesimal generator of the strongly continuous semigroup $(S(t))_{t\geq 0}$ on H and it holds*

$$\int_0^T t^{-2\theta} \|S(t)\|_{\mathcal{L}_{\mathrm{HS}}(H)}\, dt < \infty$$

for a parameter $\theta \in (0, 1/2)$.

- *The operator $B : [0,T] \times H \to \mathcal{L}(U,H)$ is measurable.*

- *The operator B satisfies a Lipschitz condition and a linear growth condition, i.e. there exists a constant $c > 0$ such that*

$$\|B(t,h) - B(t,g)\|_{\mathcal{L}(U,H)} \leq c \cdot \|h - g\|_H$$

and

$$\|B(t,h)\|_{\mathcal{L}(U,H)} \leq c \cdot (1 + \|h\|_H)$$

for every $t \in [0,T]$ and $h, g \in H$.

Now, we define a so-called mild solution for the problem (2.9) in both of the mentioned cases.

Definition 2.3.1 ***(Mild solution)***
An H-valued predictable process $(X(t))_{t\in[0,T]}$ is called a mild solution of (2.9) if

$$P\left(\int_0^T \|X(s)\|_H\, ds < \infty\right) = 1$$

and

$$P\left(\int_0^T \|B(s, X(s))\|_{\mathcal{L}}^2\, ds < \infty\right) = 1,$$

where $\mathcal{L} = \mathcal{L}_{\mathrm{HS}}^0$ in the (TC) case and $\mathcal{L} = \mathcal{L}(U,H)$ in the (ID) case, and

$$X(t) = S(t)\xi + \int_0^t S(t-s)B(s, X(s))\, dW(s)$$

P-almost surely for every $t \in [0,T]$.

We give the following results about the existence and uniqueness of mild solutions of the stochastic partial differential equation (2.9) for both of the cases (TC) and (ID).

Proposition 2.3.1 *Assume that Assumption 2.3.1 is satisfied. Then there exists a mild solution $X = (X(t))_{t\in[0,T]}$ of (2.9) in the (TC) case, which is, up to equivalence, unique among the processes satisfying*

$$P\left(\int_0^T \|X(t)\|_H^2\, dt < \infty\right). \tag{2.10}$$

Up to equivalence means here that if there exists another mild solution $\widehat{X} = (\widehat{X}(t))_{t\in[0,T]}$ of (2.9) satisfying (2.10), then $P(X(t) = \widehat{X}(t)) = 1$ for every $t \in [0,T]$. Moreover, the mild solution X has a continuous modification $\widetilde{X} = (\widetilde{X}(t))_{t\in[0,T]}$, that means $P(X(t) = \widetilde{X}(t)) = 1$ for every $t \in [0,T]$. Also, for every $p \geq 2$ there exists a constant $c_{p,T} > 0$, only depending on p and T, such that

$$\sup_{t\in[0,T]} \mathrm{E}\|X(t)\|_H^p \leq c_{p,T} \cdot (1 + \|\xi\|_H^p)\,.$$

Proof: See, e.g., Theorem 7.4 in [DPZ92]. □

Proposition 2.3.2 *Assume that Assumption 2.3.2 is satisfied. Then there exists an, up to equivalence, unique continuous mild solution $X = (X(t))_{t\in[0,T]}$ of (2.9) in the (ID) case. Moreover, for every $p \geq 2$ there exists a constant $c_{p,T} > 0$, only depending on p and T, such that*

$$\sup_{t\in[0,T]} \mathrm{E}\|X(t)\|_H^p \leq c_{p,T} \cdot (1 + \|\xi\|_H^p)\,.$$

Proof: See, e.g., Theorem 7.6 in [DPZ92]. □

2.4 Examples

In this section, we give examples for the operators A and B in the stochastic evolution equation with additive noise

$$\begin{aligned} dX(t) &= AX(t)\, dt + B(t)\, dW(t), \\ X(0) &= \xi, \end{aligned} \tag{2.11}$$

satisfying the assumptions we consider in our results.

For fixed $d \in \mathbb{N}$ let $H = L_2\left((0,1)^d\right)$ be the separable real Hilbert space of equivalence classes of square integrable functions mapping $(0,1)^d$ to $\mathbb{R}$ and $(h_j)_{j\in\mathbb{N}^d}$ be the orthonormal basis of H given by

$$h_j(u) = 2^{d/2} \cdot \prod_{\ell=1}^{d} \sin(j_\ell \cdot \pi \cdot u_\ell), \quad u \in (0,1)^d.$$

Consider as the operator $A : D(A) \subset H \to H$ the weak differential operator of the form

$$Ah = \sum_{\ell=1}^{d} \frac{\partial^\alpha}{\partial u_\ell^\alpha} h, \quad h \in D(A),$$

with order $\alpha \in 4 \cdot \mathbb{N}_0 + 2$, i.e. for $\alpha = 2$ the operator A is the Laplace operator Δ introduced in Example B.0.1 in Appendix B. Then it holds

$$Ah_j = -\mu_j \cdot h_j$$

with eigenvalues given by

$$\mu_j = \pi^\alpha \cdot |j|_2^\alpha,$$

with respect to the Euclidean norm $|\cdot|_2$. The calculation of the generated strongly continuous semigroup $(S(t))_{t\geq 0}$ is analogue to the one for $\alpha = 2$. In the case $A = \Delta$, we call (2.11) a stochastic heat equation with additive noise because for $B = 0$ we just obtain the deterministic heat equation.

Consider as the operator B a pointwise multiplication operator, i.e.

$$B(t)h = G(t) \cdot h$$

with $h \in H$ and $t \in [0,T]$, where $G : [0,T] \to H$ should satisfy the following condition. For simplicity, we write $\mathcal{G}(t,u) = G(t)(u)$ and suppose that

$$\mathcal{G} \in C^{(1,1,\ldots,1)}([0,T] \times [0,1]^d).$$

We set

$$B_{ij}(t) = \langle B(t)h_i, h_j \rangle_H = \int_{(0,1)^d} \mathcal{G}(t,u) \cdot h_i(u) \cdot h_j(u)\, du, \quad t \in [0,T],$$

and

$$\delta_{ij} = \begin{cases} \prod_{\substack{\ell=1 \\ i_\ell \neq j_\ell}}^{d} |i_\ell - j_\ell|^{-1}, & \text{if } i \neq j, \\ 1, & \text{if } i = j, \end{cases}$$

for $i, j \in \mathbb{N}^d$. Then it holds $B_{ij} \in C^1([0,T])$ and

$$\sup_{t \in [0,T]} \left(|B_{ij}(t)|^2 + |B'_{ij}(t)|^2 \right) \leq c_d \cdot \delta_{ij}^2 \tag{2.12}$$

with a constant $c_d > 0$, which only depends on the parameter d. For the proofs and more details, see [MGR07a]. Moreover, we can use the Lemma of Lax-Milgram, stated, e.g., in Chapter 5 in [W07], to see that there exist time-constant operators $B \in \mathcal{L}(H)$, such that the term on the left hand side in (2.12) can be expressed by

$$|B_{ij}|^2 = \delta_{ij}^\beta$$

with a fixed $\beta \geq 2$. To see this, we prove the following lemma.

Lemma 2.4.1 *Let $d \in \mathbb{N}$. For every $p \geq 1$ and every orthonormal basis $(h_j)_{j \in \mathbb{N}^d}$ of a separable Hilbert space H there exists an operator $B \in \mathcal{L}(H)$ such that*

$$\delta_{ij}^p = \langle Bh_i, h_j \rangle_H$$

for every $i, j \in \mathbb{N}^d$.

Proof: Define

$$\mathcal{B}_p(g,h) = \sum_{k \in \mathbb{N}^d} \langle g, h_k \rangle_H^2 \cdot \sum_{\ell \in \mathbb{N}^d} \langle h, h_\ell \rangle_H^2 \cdot \delta_{k\ell}^p$$

for $g, h \in H$. Thus,

$$\mathcal{B}_p(h_i, h_j) = \delta_{ij}^p$$

for $i, j \in \mathbb{N}^d$ and

$$\begin{aligned} |\mathcal{B}_p(g,h)| &\leq \sum_{k \in \mathbb{N}^d} \langle g, h_k \rangle_H^2 \cdot \sum_{\ell \in \mathbb{N}^d} \langle h, h_\ell \rangle_H^2 \\ &\leq \|g\|_H \cdot \|h\|_H \end{aligned}$$

using the Bessel inequality. Hence, by the lemma of Lax-Milgram there exists a mapping $B \in \mathcal{L}(H)$ such that $\langle Bg, h \rangle_H = \mathcal{B}_p(g,h)$ for every $g, h \in H$ and the claim follows. □

2.5 Survey of Known Approximation Results

In this section we briefly overview some known results about the numerical approximation for stochastic evolution equations in the literature. Here we can only give a rough summary because of the large number of achievements in this topic in recent years. We refer to the cited articles and the references therein for further results.

One of the first algorithms for a parabolic stochastic partial differential equation with Dirichlet boundary conditions on a bounded domain $\mathcal{D}$ in $\mathbb{R}^d$ is given in [GK96]. In this paper the equation is of the form

$$dX(t) = (AX(t) + f(X(t)))\,dt + B(X(t))\,dW(t), \tag{2.13}$$

where the process W is considered as a scalar Brownian motion. Furthermore, the authors assume that the eigenfunctions $(h_i)_{i\in\mathbb{N}}$ of the linear operator $-A$ with the corresponding eigenvalues $(\mu_i)_{i\in\mathbb{N}}$ form an orthonormal basis of $L_2(\mathcal{D})$ where $h_i \in H^2(\mathcal{D})\cap H_0^1(\mathcal{D})$ and $\mu_i \to \infty$ as $i \to \infty$. The authors show that the global discretization error for a stochastic Taylor scheme $\widehat{X}_k^N$ of strong order γ with constant time-step Δ applied to an N-dimensional Ito-Galerkin equation corresponding to (2.13) is of the form

$$\mathrm{E}\left(\left|X(k\Delta) - \widehat{X}_k^N\right|_{L_2(\mathcal{D})}\right) \leq C\cdot\left(\mu_{N+1}^{-1/2} + \mu_N^{\lfloor\gamma+1/2\rfloor+1}\cdot\Delta^\gamma\right).$$

In this estimate, $\lfloor x\rfloor$ denotes the integer part of the real number x and the positive constant C only depends on the initial value, the coefficient functions and on the length of the time intervall $0 \leq k\Delta \leq T$. This result could be improved in [KS01] by using a drift-implicit stochastic Taylor scheme $\widehat{X}_k^N$ of strong order γ such that the error is of the form

$$\mathrm{E}\left(\left|X(k\Delta) - \widehat{X}_k^N\right|_{L_2(\mathcal{D})}\right) \leq C\cdot\left(\mu_{N+1}^{-1/2} + \Delta^\gamma\right).$$

For instance, considering the drift-implicit Euler-Maruyama scheme $\widetilde{X}_k^M$ with an equidistant time discretization based on N evaluations of the driving scalar Brownian motion, it holds

$$\mathrm{E}\left(\left|X(k\Delta) - \widetilde{X}_k^N\right|_{L_2(\mathcal{D})}\right) \leq C\cdot N^{-1/2}$$

in the case that μ_i is proportional to i^2.

In [GN95] the authors consider the semilinear stochastic heat equation

$$dX(t) = (\Delta X(t) + f(X(t)))\,dt + dW(t) \tag{2.14}$$

with additive space-time white noise on the one-dimensional domain $[0,1]$ over the time interval $[0,T]$ with $T > 0$. They introduce an implicit approximation scheme, which converges uniformly in probability to the exact solution. In [S99] the author applies a finite difference scheme to the above equation to obtain a discretization in space. Then, he provides a method of time discretization for the resulting finite dimensional coupled system of equations. He shows for an approximation $\widehat{X}_N(T)$ a convergence order of $1/6 - \epsilon$ for every $\epsilon > 0$ with respect to the number N of evaluations of the driving cylindrical Wiener process, i.e.

$$\left(\mathrm{E}\|X(T) - \widehat{X}_N(T)\|_H^2\right)^{1/2} \leq C \cdot N^{-1/6+\epsilon}.$$

In the articles [G98] and [G99], for a stochastic heat equation with multiplicative noise the author also substitutes the space derivatives with a finite difference method and then uses temporal explicit and implicit schemes, i.e. the implicit Euler method. For a smooth initial value, those schemes converge with rate $1/2$ in space and with rate $1/4$ in time. Therefore, an overall order of convergence of $1/6$ is established with respect to the number of evaluations in space and time. In [JK09a] the authors present the so-called exponential Euler scheme for the equation (2.14) to exceed this rate. It uses suitable linear functionals of the noise and achieves the improved convergence order of $1/3$. It turns out that any approximation scheme applied to the equation (2.14) with $f = 0$ that only uses equidistant values of the driving Wiener process W cannot exceed the convergence rate of $1/6$. This can be shown by estimating lower error bounds.

In [DG01] first results are stated about lower error bounds for the strong approximation of an equation of the form (2.13) in the space-time white noise case. For linear equations, i.e. $f = 0$, with a specific multiplicative noise the authors prove that any approximation scheme using equidistant values of the noise W has at most the order of convergence $1/6$ with respect to the noise evaluations. In [MGR07a] the authors consider the stochastic heat equation

$$dX(t) = \Delta X(t)\,dt + B(t, X(t))\,dW(t) \tag{2.15}$$

on the Hilbert space $H = L_2((0,1)^d)$ in the nuclear noise as well as in the space-time white noise case. The multiplicative noise is given by pointwise multiplication $B(t,x)h = G(t,x)\cdot h$ for $x, h \in H$ and $t \geq 0$ with $G : [0,T] \times H \to H$ satisfying mild regularity conditions. Considering the global error

$$\mathrm{e}\left(\widehat{X}_N\right) = \left(\mathrm{E}\int_0^T \|X(t) - \widehat{X}_N(t)\|_H^2\, dt\right)^{1/2}$$

in space and time of an approximation $\widehat{X}_N$ based on N evaluations of the scalar components of the driving Wiener process, the Nth minimal error

$$\mathrm{e}_N = \inf_{\widehat{X}_N} \mathrm{e}\left(\widehat{X}_N\right)$$

has the lower bounds

$$\mathrm{e}_N \geq C \cdot N^{-1/6} \tag{2.16}$$

in the (ID) case and

$$\mathrm{e}_N \geq C \cdot \begin{cases} N^{-1/2+(d-\gamma/2)/(d+2)}, & \text{if } d < \gamma < 2d, \\ N^{-1/2}\cdot \ln N, & \text{if } \gamma = 2d, \\ N^{-1/2}, & \text{if } \gamma > 2d, \end{cases} \tag{2.17}$$

in the (TC) case. Here C is a positive constant only depending on the equation and γ controls the smoothness of the noise where larger values of γ lead to a higher smoothness. Furthermore, for the equation (2.15) with additive noise the authors construct asymptotically optimal algorithms that achieve the rates of convergence obtained in (2.16) and (2.17). The presented schemes base on an equidistant but non-uniform time discretization of W.

In [MGRW08] the authors consider the equation (2.15) with the specific additive noise $B(t,x) = I$ where I is the identity operator on H and study the pointwise error

$$\mathrm{e}\left(\widehat{X}_N(T)\right) = \left(\mathrm{E}\|X(T) - \widehat{X}_N(T)\|_H^2\right)^{1/2}$$

of any approximation scheme $\widehat{X}_N$ at time point $T > 0$ that again uses N evaluations of the scalar components of the driving Wiener process W. In this paper, it is proven for the corresponding Nth minimal error

$$\mathrm{e}_N \geq C \cdot N^{-1/2} \tag{2.18}$$

in the (ID) case and

$$\mathrm{e}_N \geq C \cdot \begin{cases} N^{-(\gamma-2+2)/(2d)}, & \text{if } d < \gamma < 3d-2, \\ N^{-1} \cdot (\ln N)^{3/2}, & \text{if } \gamma = 3d-2, \\ N^{-1}, & \text{if } \gamma > 3d-2, \end{cases} \tag{2.19}$$

in the (TC) case with a positive constant C only depending on the equation and the smoothness parameter γ for the noise. Moreover, asymptotically optimal algorithms, which achieve the rates (2.18) and (2.19) are presented. This schemes base on drift-implicit Euler-Maruyama schemes using non-uniform and even non-equidistant time discretization. The analysis of the respective Nth minimal error shows that asymptotic optimality cannot be achieved by algorithms with equidistant time discretization in the (ID) case and for $\gamma < 3d-2$ in the (TC) case. Hence, in contrast to the results for the global error criterion, the non-equidistant time discretization is superior to all the equidistant ones in case of space-time white noise and nuclear noise with smaller smoothness.

In this work we extend the results of [MGRW08] by considering a stochastic evolution equation with more general operators in the drift and diffusion term.

Chapter 3

Approximation of Systems of Ornstein-Uhlenbeck Equations

In this chapter we consider the following stochastic evolution equation

$$\begin{aligned} dX(t) &= AX(t)\,dt + B(t)\,dW(t), \quad t \in [0,T], \\ X(0) &= \xi, \end{aligned} \tag{3.1}$$

with additive noise on a compact time interval with $T > 0$. We either study this equation with nuclear noise or space-time white noise on the real Hilbert space $H = L_2\left((0,1)^d\right)$ for a fixed $d \in \mathbb{N}$. Throughout this chapter $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and the scalar product in H, and we distinguish between the two cases of nuclear noise and space-time white noise, shortly called (TC) and (ID), respectively. In order to formulate assumptions for the objects of the equation (3.1) we introduce the following notation for convenience.

Definition 3.0.1 *Let $\mathcal{N}$ be a countable index set and let $(x_N)_{N\in\mathcal{N}}$, $(y_N)_{N\in\mathcal{N}}$ be two sequences of positive real numbers. We write*

$$x_N \preceq y_N, \text{ if } \sup_{N\in\mathcal{N}} \frac{x_N}{y_N} < \infty$$

and call x_N weakly asymptotically smaller than y_N. Moreover, we write

$$x_N \asymp y_N, \text{ if } x_N \preceq y_N \text{ and } y_N \preceq x_N,$$

and call x_N weakly asymptotically equal to y_N.

Hence, the objects of the equation should fulfil the following conditions.

Assumption 3.0.1 ***(Wiener process*** W***)***
Let $(h_j)_{j\in\mathbb{N}^d}$ *be an orthonormal basis of* H *and let* $(\Omega, \mathcal{F}, P)$ *be a complete probability space with a right continuous filtration* $(\mathcal{F}_t)_{t\in[0,T]}$*.*

(TC) The process $W = (W(t))_{t\in[0,T]}$ *is a* Q*-Wiener process on* H *with a trace class covariance operator* $Q : H \to H$*. Furthermore, the basis* $(h_j)_{j\in\mathbb{N}^d}$ *is a sequence of eigenfunctions of* Q *with the corresponding eigenvalues*

$$\lambda_j \asymp |j|_2^{-\gamma} \tag{3.2}$$

for every $j \in \mathbb{N}^d$ *with respect to the Euclidean norm* $|\cdot|_2$ *and* $\gamma > d$*.*

(ID) The process $W = (W(t))_{t\in[0,T]}$ *is a cylindrical Wiener process on* H *with the covariance operator* $Q = I$*, where* I *is the identity operator on* H*. Furthermore, it holds* $d = 1$*.*

In this Assumption 3.0.1 as well as in the following ones, we use the index set $\mathbb{N}^d$ for notational convenience instead of, for instance, the conventional choice $\mathbb{N}$, which is isomorph. Note that we have

$$Qh = \sum_{j\in\mathbb{N}^d} \lambda_j \cdot \langle h, h_j\rangle \cdot h_j$$

for every $h \in H$ with

$$\sum_{j\in\mathbb{N}^d} \lambda_j < \infty.$$

in the (TC) case and

$$\lambda_j = 1$$

for every $j \in \mathbb{N}$ in the (ID) case, which implies the setting $\gamma = 0$ in (3.2). In particular, by changing the parameter γ we influence the speed of the decay of the eigenvalues of the covariance operator Q. That means that the smoothness of the noise and the smoothness of the solution X, too, is controlled by γ and larger values of γ lead to higher smoothness.

In the following assumptions, let $\mathcal{L}(H) = \mathcal{L}(H, H)$ be the class of all bounded linear operators from H to H equipped with the operator norm $\|\cdot\|_{\mathcal{L}(H)}$ and let $\mathcal{L}_{\mathrm{HS}}(H) = \mathcal{L}_{\mathrm{HS}}(H, H)$ be the class of all Hilbert-Schmidt operators from H into H equipped with the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$. Furthermore, we define for the (TC) case the Hilbert space

$$H_0 = Q^{1/2} H$$

with respect to the scalar product

$$\left\langle Q^{1/2} h_1, Q^{1/2} h_2 \right\rangle_{H_0} = \langle h_1, h_2 \rangle .$$

Recall from Chapter 2, that in this case Q is a bounded linear nonnegative symmetric nuclear operator and therefore $(\lambda_j^{1/2} \cdot h_j)_{j \in \mathbb{N}^d}$ is an orthonormal basis of H_0. Moreover, let $\mathcal{L}_{\mathrm{HS}}^0 = \mathcal{L}_{\mathrm{HS}}(H_0, H)$ be the class of Hilbert-Schmidt operators from H_0 into H equipped with the Hilbert-Schmidt norm $\|\cdot\|_{\mathcal{L}_{\mathrm{HS}}^0}$ and the Borel σ-algebra $\mathcal{B}(\mathcal{L}_{\mathrm{HS}}^0)$. In the (ID) case we use the smallest σ-algebra $\mathcal{S}$ of subsets of $\mathcal{L}(H)$ containing all sets of the form $\{\Lambda \in \mathcal{L}(H) \,|\, \Lambda h \in \mathcal{H}\}$ with $h \in H$ and $\mathcal{H} \in \mathcal{B}(H)$.

Assumption 3.0.2 ***(Diffusion term*** B***)***

(TC) The mapping

$$B : [0, T] \to \mathcal{L}_{\mathrm{HS}}^0$$

is measurable from $([0, T], \mathcal{B}([0, T]))$ *into* $(\mathcal{L}_{\mathrm{HS}}^0, \mathcal{B}(\mathcal{L}_{\mathrm{HS}}^0))$ *and there exists a constant* $c > 0$*, such that*

$$\|B(t)\|_{\mathcal{L}_{\mathrm{HS}}^0} \leq c$$

for every $t \in [0, T]$*.*

(ID) The mapping

$$B : [0, T] \to \mathcal{L}(H)$$

is measurable from $([0, T], \mathcal{B}([0, T]))$ *into* $(\mathcal{L}(H), \mathcal{S})$ *and there exist a constant* $c > 0$*, such that*

$$\|B(t)\|_{\mathcal{L}(H)} \leq c$$

for every $t \in [0, T]$*.*

In both cases, with $\mathcal{L} = \mathcal{L}_{\mathrm{HS}}^0$ in the (TC) case and $\mathcal{L} = \mathcal{L}(H)$ in the (ID) case, it holds

$$\int_0^T \|B(t)\|_{\mathcal{L}}^2 \, dt > 0$$

to exclude deterministic equations and

$$t \mapsto \langle B(t)h_i, h_j \rangle \in C^1([0,T])$$

for every $i, j \in \mathbb{N}^d$, where h_i and h_j are basis functions of the orthonormal basis introduced in Assumption 3.0.1. Furthermore, it holds

$$\inf_{t\in[0,T]} \langle B(t)h_i, h_i \rangle^2 \succeq 1 \tag{3.3}$$

and

$$\sup_{t\in[0,T]} \langle B(t)h_i, h_j \rangle^2 \preceq \begin{cases} \prod\limits_{\substack{\ell=1 \\ i_\ell \neq j_\ell}}^{d} |i_\ell - j_\ell|^{-\beta}, & \text{if } i \neq j, \\ 1, & \text{if } i = j, \end{cases} \tag{3.4}$$

for every $i, j \in \mathbb{N}^d$ and a fixed parameter $\beta > 1$.

The parameter β in the Assumption 3.0.2 controls the decay of the scalar product $\langle B(t)h_i, h_j \rangle$ for different values of i and j while moving away from the diagonal elements. Hence, larger values of β lead to a higher decoupling between different space dimensions of H by $B(t)$. For $\beta = 2$, the operator $B(t)$ corresponds to a pointwise multiplication operator and even for $\beta > 2$ there exist operators, which fulfil (3.4). See Section 2.4 for more details and an example.

Assumption 3.0.3 ***(Generator A and initial value ξ)***
The eigenfunctions $(h_j)_{j\in\mathbb{N}^d}$ of Q are also eigenfunctions of the linear operator $A : D(A) \subset H \to H$, which is given by

$$Ah = \sum_{j\in\mathbb{N}^d} -\mu_j \cdot \langle h, h_j \rangle \cdot h_j$$

for every $h \in D(A) = \Big\{ h \in H \;\Big|\; \sum_{j\in\mathbb{N}^d} |\mu_j|^2 \cdot |\langle h, h_j \rangle|^2 < \infty \Big\}$. The negative eigenvalues of A are of the form

$$\mu_j \asymp |j|_2^\alpha \tag{3.5}$$

for every $j \in \mathbb{N}^d$ and a fixed exponent $\alpha \geq 2$.

The initial value $\xi \in D(A)$ is assumed to be deterministic.

Note that $D(A)$ is dense in H and furthermore that A is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on H with

$$S(t)h = \sum_{j\in\mathbb{N}^d} \exp(-\mu_j t) \cdot \langle h, h_j\rangle \cdot h_j$$

for arbitrary $h \in H$ and $t \geq 0$. Moreover, it holds

$$\|S(t)\|_{\mathrm{HS}}^2 = \sum_{j\in\mathbb{N}^d} \exp(-2\mu_j t). \tag{3.6}$$

For more details, see, e.g., Chapter II.3 in [EN00], i.e., the Hille-Yosida Theorem 3.5. In the case that $\alpha = 2$ the generator A corresponds to the Laplace operator Δ, which is introduced in Example B.0.1. Additionally, we need a further assumption on the semigroup $(S(t))_{t\geq 0}$ in the (ID) case.

Assumption 3.0.4 ***(Semigroup in the (ID) case)***
In the (ID) case, for a parameter $\theta \in (0, 1/2)$ it holds

$$\int_0^T t^{-2\theta} \|S(t)\|_{\mathrm{HS}}^2 \, dt < \infty \tag{3.7}$$

where $(S(t))_{t\geq 0}$ is the semigroup on H generated by A.

With this Assumption 3.0.4 we are able to explain why we only consider $d = 1$ in the (ID) case.

Remark 3.0.1 ***(Restriction $d = 1$ in the (ID) case)***
If we consider the eigenvalues of the operator A in the drift term of the form $\mu_j \asymp |j|_2^\alpha$, with $\alpha \geq 2$, as we do, then the setting $d = 1$ ensures that the inequality (3.7) in Assumption 3.0.4 is fulfilled. To see this, we put for convenience $T = 1$ and use with $\theta \in (0, 1/2)$ the estimate

$$\begin{aligned} \int_0^1 t^{-2\theta} \exp(-2\mu_j t)\, dt &\leq \int_0^{1/j^\alpha} t^{-2\theta}\, dt + \left(\max_{1/j^\alpha \leq t \leq 1} t^{-2\theta}\right) \cdot \int_{1/j^\alpha}^1 \exp(-2\mu_j t)\, dt \\ &\leq \frac{1}{1-2\theta} \cdot j^{\alpha(2\theta-1)} + j^{2\alpha\theta} \cdot \frac{1}{2\mu_j}(\exp(-2\mu_j/j^\alpha) - \exp(-2\mu_j)) \\ &\preceq \frac{1}{1-2\theta} \cdot j^{\alpha(2\theta-1)} + j^{\alpha(2\theta-1)}. \end{aligned}$$

Thus, by (3.6), the condition (3.7) holds for $d = 1$ and $\theta \in (0, (\alpha-1)/(2\alpha))$. Otherwise, if $d \in \mathbb{N} \setminus \{1\}$, we have

$$\int_0^1 \|S(t)\|_{\mathrm{HS}}^2 \, dt \asymp \sum_{j \in \mathbb{N}^d} |j|_2^{-\alpha} \cdot (1 - \exp(-2\mu_j)) \succeq \int_1^\infty r^{-\alpha+d-1} \, dr$$

using (3.6) and Lemma C.0.3. Thus, the condition (3.7) does not even hold for $\theta = 0$ if $\alpha \leq d$, which includes the important special case $\alpha = 2$. $\diamond$

Some of our statements additionally use the assumption that $\langle \xi, h_j \rangle^2 \preceq \lambda_j$ for every $j \in \mathbb{N}^d$. Clearly, this always holds true if $\xi = 0$ and also if $\xi \in H$ in the (ID) case. In the (TC) case this describes a smoothness condition for ξ.

We know from Chapter 2, that under the Assumptions 3.0.1 to 3.0.4 in both cases (TC) and (ID) the mild solution $(X(t))_{t \in [0,T]}$ of (3.1) is a continuous process with values in H and

$$X(t) = S(t)\xi + \int_0^t S(t-s)B(s) \, dW(s) \tag{3.8}$$

holds P-almost surely for every $t \in [0, T]$. Also, this process is uniquely determined P-almost surely and it satisfies $\sup_{t \in [0,T]} \mathrm{E}\|X(t)\|^p < \infty$ for every $p \geq 2$. We put

$$\beta_i(t) = \lambda_i^{-1/2} \langle W(t), h_i \rangle$$

for every $i \in \mathbb{N}^d$ and $t \in [0, T]$ to get an independent family of standard one-dimensional Brownian motions $(\beta_i)_{i \in \mathbb{N}^d}$ as a spatial discretization of the Wiener process W in H. Then, by Assumptions 3.0.1 to 3.0.4, the Fourier expansion

$$X(t) = \sum_{j \in \mathbb{N}^d} \left(\exp(-\mu_j t) \cdot \langle \xi, h_j \rangle + \sum_{i \in \mathbb{N}^d} \lambda_i^{1/2} \cdot Z_{ij}(t) \right) \cdot h_j \tag{3.9}$$

of the mild solution with respect to the basis functions $(h_j)_{j \in \mathbb{N}^d}$ holds P-almost surely in H and $L_2(\Omega, \mathcal{F}, P; H)$ for $t \in [0, T]$. Here we use the scalar stochastic processes

$$Z_{ij}(t) = \int_0^t \exp(-\mu_j(t-s)) \cdot \langle B(s)h_i, h_j \rangle \, d\beta_i(s) \tag{3.10}$$

for $i, j \in \mathbb{N}^d$. Note that the $\mathbb{R}$-valued stochastic process $(Z(t))_{t\geq 0}$ satisfying the ordinary stochastic differential equation

$$\begin{aligned} dZ(t) &= c \cdot (c_1 - Z(t))\, dt + k\, d\beta(t), \quad t \geq 0, \\ Z(0) &= c_0, \end{aligned}$$

is given by

$$Z(t) = c_0 \cdot \exp(-ct) + c_1 \cdot (1 - \exp(-ct)) + \int_0^t k \cdot \exp(-c(t-s))\, d\beta(s), \quad t \geq 0,$$

with constants $c > 0$, $k, c_0, c_1 \in \mathbb{R}$ and a scalar Brownian motion $(\beta(t))_{t\geq 0}$. It is called Ornstein-Uhlenbeck process on $\mathbb{R}$. The processes $(Z_{ij})_{i,j\in\mathbb{N}^d}$ form a family of possibly coupled Ornstein-Uhlenbeck processes on $\mathbb{R}$, if we have a time constant scalar product $\langle Bh_i, h_j\rangle$ for every $i, j \in \mathbb{N}^d$. Therefore, we call the mild solution (3.9) an Ornstein-Uhlenbeck process on H.

In the next sections, we introduce the classes of algorithms considered to approximate the mild solution X at the fixed time point T, as well as the error criterion and costs of these approximations. Following, we construct and analyze algorithms and state results about their quality by comparing its error and cost. The proofs of the results in this chapter can be found in Section 3.4.

3.1 Classes of Algorithms

We approximate the mild solution X of (3.1) at the time point $T > 0$. For this purpose we study algorithms, which evaluate a finite number of the scalar stochastic processes β_i, $i \in \mathbb{N}^d$, used in (3.10), at a finite number of time points. By this approach we can establish different approximation schemes, which use, respectively, different space discretizations of the noise W. Furthermore, for the evaluation, in the chosen space dimensions different time discretizations may be considered.

Formally this means, with an arbitrary $k \in \mathbb{N}$, we specify an index set

$$\mathcal{I} = \{i_1, \ldots, i_k\} \subset \mathbb{N}^d,$$

a finite sequence

$$\mathfrak{n} = (n_i)_{i\in\mathcal{I}} \in \mathbb{N}^{\mathcal{I}}$$

of integers and time nodes

$$0 = t_{0,i} < t_{1,i} < \cdots < t_{n_i,i} \leq T \tag{3.11}$$

for every $i \in \mathcal{I}$. We call a family $(t_{k,i})_{k=0,\ldots,n_i,i\in\mathcal{I}}$ of time nodes defined by (3.11) a space-time discretization of W. Now, every one-dimensional Brownian motion β_i with $i \in \mathcal{I}$ is evaluated at the respective time nodes $(t_{k,i})_{k=1,\ldots,n_i}$. So, the total number of evaluations is given by

$$|\mathfrak{n}|_1 = \sum_{i\in\mathcal{I}} n_i.$$

An approximation $\widehat{X}(T)$ of $X(T)$ is formally defined by

$$\widehat{X}(T) = \phi\left(\beta_{i_1}(t_{1,i_1}), \ldots, \beta_{i_1}(t_{n_{i_1},i_1}), \ldots, \beta_{i_k}(t_{1,i_k}), \ldots, \beta_{i_k}(t_{n_{i_k},i_k})\right) \tag{3.12}$$

with a measurable mapping

$$\phi : \mathbb{R}^{|\mathfrak{n}|_1} \to H.$$

For $N \in \mathbb{N}$, let $\mathfrak{X}_N^*$ denote the class of all algorithms (3.12) that use at most a total of N evaluations of the scalar Brownian motions $(\beta_i(t))_{t\in[0,T]}$ with $i \in \mathbb{N}^d$, i.e. $|\mathfrak{n}|_1 \leq N$.

Furthermore, we consider two different subclasses of $\mathfrak{X}_N^*$, denoted by $\mathfrak{X}_N^{\mathrm{equi}}$ and $\mathfrak{X}_N^{\#}$. The first one, $\mathfrak{X}_N^{\mathrm{equi}}$, consists of all approximations $\widehat{X}(T) \in \mathfrak{X}_N^*$ that use equidistant time nodes to evaluate the scalar Brownian motions $(\beta_i(t))_{t\in[0,T]}$ with $i \in \mathbb{N}^d$, i.e. $|\mathfrak{n}|_1 \leq N$ and $t_{k,i} = k/n_i \cdot T$, $k = 0, \ldots, n_i$, for every $i \in \mathbb{N}^d$. The second one, $\mathfrak{X}_N^{\#}$, consists of all approximations $\widehat{X}(T) \in \mathfrak{X}_N^*$ that use the same number of time nodes to evaluate the scalar Brownian motions $(\beta_i(t))_{t\in[0,T]}$ with $i \in \mathbb{N}^d$, i.e. $n_i = n$ with $n \in \mathbb{N}$ for every $i \in \mathbb{N}^d$ and $|\mathfrak{n}|_1 = n \cdot |\mathcal{I}| \leq N$.

At last, let $\mathfrak{X}_N^{\mathrm{uni}} = \mathfrak{X}_N^{\mathrm{equi}} \cap \mathfrak{X}_N^{\#}$ denote the subclass of all such approximations $\widehat{X}(T) \in \mathfrak{X}_N^*$ that use the same number of equidistant time nodes for every one of the scalar Brownian motions $(\beta_i(t))_{t\in[0,T]}$, i.e. $n = n_i$ and $t_{k,i} = k/n \cdot T$, $k = 0, \ldots, n$, for every $i \in \mathbb{N}^d$ and some $n \in \mathbb{N}$ with $|\mathfrak{n}|_1 = n \cdot |\mathcal{I}| \leq N$. Such a time discretization is called an uniform time discretization of W.

The error of an approximation $\widehat{X}(T)$ is defined by

$$\mathrm{e}(\widehat{X}(T)) = \left(\mathrm{E}\|X(T) - \widehat{X}(T)\|^2\right)^{1/2},$$

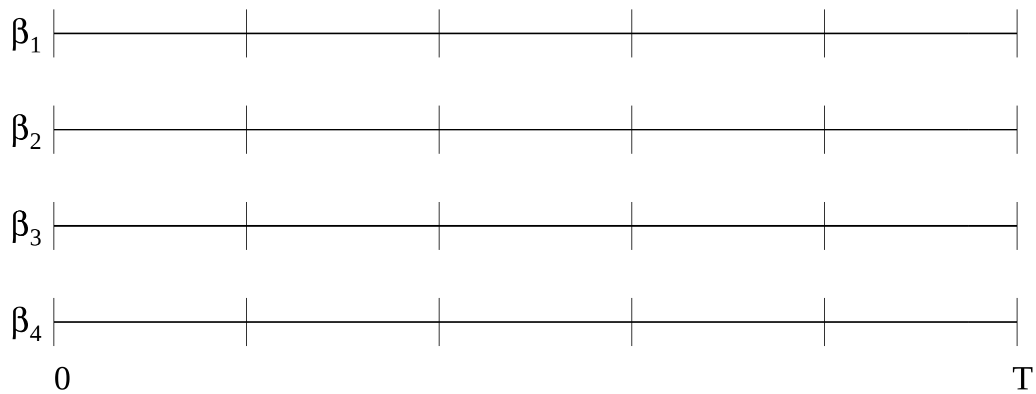

Figure 3.1: Example of a time discretization used by an algorithm in $\mathfrak{X}_N^{\mathrm{uni}}$

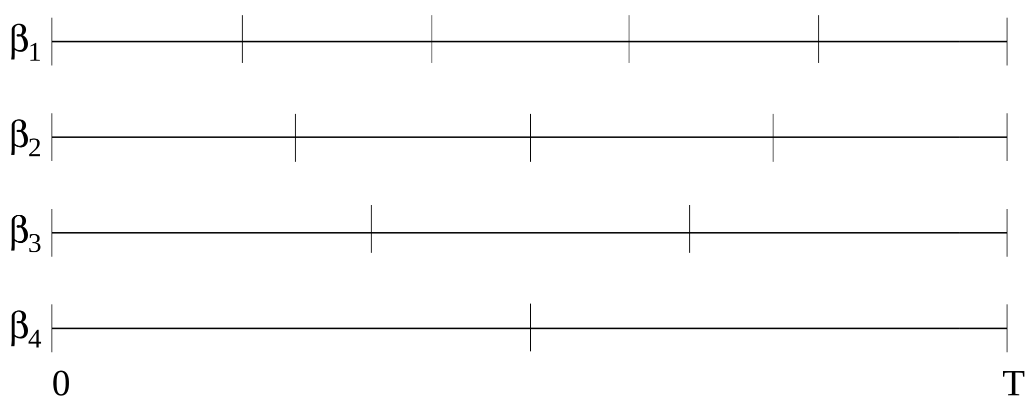

Figure 3.2: Example of a time discretization used by an algorithm in $\mathfrak{X}_N^{\mathrm{equi}}$

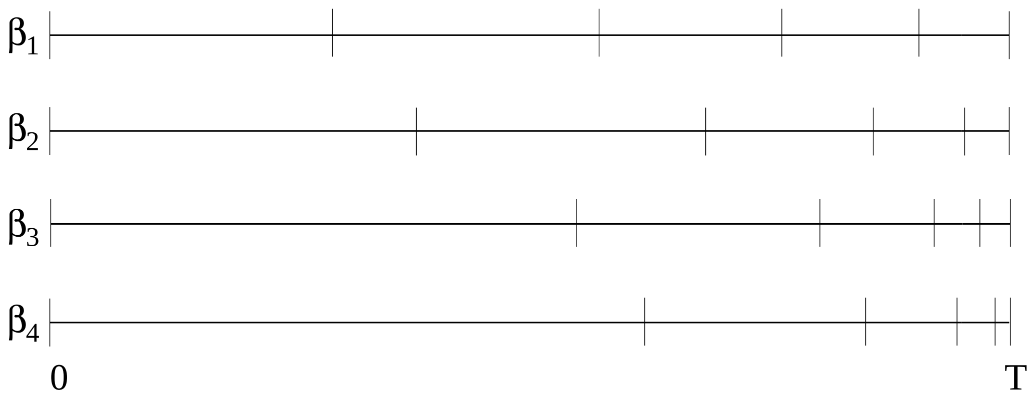

Figure 3.3: Example of a time discretization used by an algorithm in $\mathfrak{X}_N^{\#}$

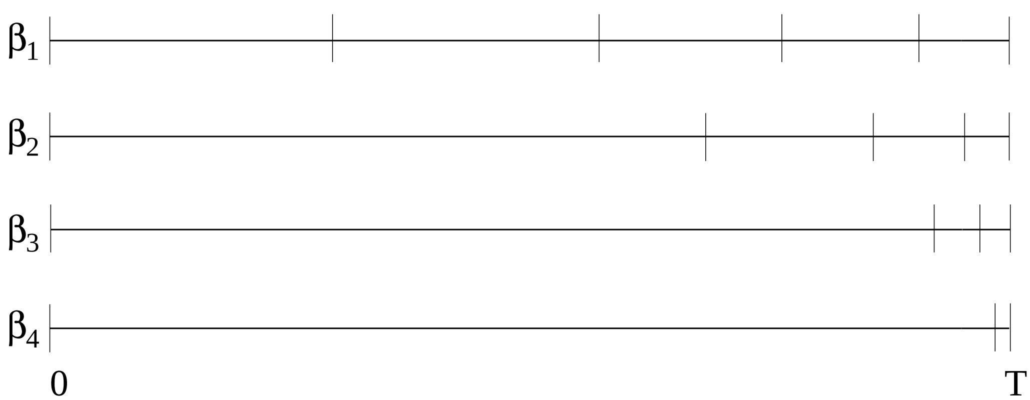

Figure 3.4: Example of a time discretization used by an algorithm in $\mathfrak{X}_N^{*}$

which describes the average distance in H between the solution and its approximation at the time point T. We are interested in algorithms, that minimize the error in the respective classes. Consequently, we study the Nth minimal errors

$$\begin{aligned} \mathrm{e}_N^* &= \inf\left\{\mathrm{e}(\widehat{X}(T)) \,|\, \widehat{X}(T) \in \mathfrak{X}_N^*\right\}, \\ \mathrm{e}_N^\# &= \inf\left\{\mathrm{e}(\widehat{X}(T)) \,|\, \widehat{X}(T) \in \mathfrak{X}_N^\#\right\}, \\ \mathrm{e}_N^{\mathrm{equi}} &= \inf\left\{\mathrm{e}(\widehat{X}(T)) \,|\, \widehat{X}(T) \in \mathfrak{X}_N^{\mathrm{equi}}\right\} \end{aligned}$$

and

$$\mathrm{e}_N^{\mathrm{uni}} = \inf\left\{\mathrm{e}(\widehat{X}(T)) \,|\, \widehat{X}(T) \in \mathfrak{X}_N^{\mathrm{uni}}\right\}.$$

As the computational cost of an approximation, we consider

$$\mathrm{cost}\left(\widehat{X}(T)\right) = |\mathfrak{n}|_1,$$

such that the single evaluation of one scalar Brownian motion is assumed to be of cost one. So, N is the upper bound for the computational cost of every algorithm $\widehat{X}(T) \in \mathfrak{X}_N^*$ and therefore $\mathrm{e}_N^{\mathrm{uni}}$, $\mathrm{e}_N^{\mathrm{equi}}$, $\mathrm{e}_N^\#$ or rather e_N^* are the smallest errors that can be achieved by any algorithm (3.12) using its respective time discretization with computational cost at most N. Immediately, it follows from the definitions, that

$$\mathrm{e}_N^* \leq \mathrm{e}_N^{\mathrm{equi}} \leq \mathrm{e}_N^{\mathrm{uni}}$$

as well as

$$\mathrm{e}_N^* \leq \mathrm{e}_N^\# \leq \mathrm{e}_N^{\mathrm{uni}},$$

because of $\mathfrak{X}_N^{\mathrm{uni}} \subset \mathfrak{X}_N^{\mathrm{equi}} \subset \mathfrak{X}_N^*$ and $\mathfrak{X}_N^{\mathrm{uni}} \subset \mathfrak{X}_N^\# \subset \mathfrak{X}_N^*$.

We want to establish error bounds for an approximation $\widehat{X}_N(T) \in \mathfrak{X}_N^*$ of the form

$$c_1 \cdot N^{-d_1} \leq \mathrm{e}(\widehat{X}_N(T)) \leq c_2 \cdot N^{-d_2}$$

with exponents $d_1, d_2 > 0$ and arbitrary constants $c_1, c_2 > 0$, which may depend on the equation, i.e. on d, $(\lambda_i)_{i\in\mathbb{N}^d}$, A, B, ξ and T, but are independent of the cost N. We call d_1 and d_2 respectively the order of convergence of the lower and the upper error bound of approximation $\widehat{X}_N(T)$ and disregard the investigation of the factors c_1 and c_2. To avoid mentioning these factors every time, we use the notation introduced in Definition 3.0.1. Of course, we wish to construct a sequence of algorithms $\widehat{X}_N(T)$ with order of convergence $d_1 = d_2$ in all of the considered classes, i.e. in weakly asymptotic notation we want to achieve

$$\mathrm{e}(\widehat{X}_N(T)) \asymp \mathrm{e}_N^{\diamond} \text{ for } \widehat{X}_N(T) \in \mathfrak{X}_N^{\diamond}.$$

with $\diamond \in \{*, \#, \mathrm{equi}, \mathrm{uni}\}$. Such sequences of algorithms are called weakly asymptotically optimal and are derived separately for systems of decoupled and coupled Ornstein-Uhlenbeck processes in the Sections 3.2 and 3.3.

Thus, the common approach in the following sections to approximate the mild solution (3.9) at T by $\widehat{X}_N^{\diamond}(T) \in \mathfrak{X}_N^{\diamond}$ for fixed cost $N \in \mathbb{N}$ and $\diamond \in \{*, \#, \mathrm{equi}, \mathrm{uni}\}$ goes as follows. We specify a non-empty finite set

$$\mathcal{I}_N \subset \mathbb{N}^d$$

as the space discretization of W and nodes

$$0 < t_{1,i} < \cdots < t_{n_i,i} \leq T$$

for $i \in \mathcal{I}_N$ and $n_i \in \mathbb{N}$ as the time discretization of W. Furthermore, we choose a second non-empty finite set

$$\mathcal{J}_N \subset \mathbb{N}^d$$

as a space discretization of the solution X. Now, we define for every combination of $j \in \mathcal{J}_N$ and $i \in \mathcal{I}_N$ an approximation scheme $\widehat{Z}_{ij,N}$, which uses the evaluated values out of the sequence $(\beta_i(t_{1,i}), \ldots, \beta_i(t_{n_i,i}))$, to estimate $Z_{ij}(T)$. Finally, we put

$$\widehat{X}_N(T) = \sum_{j \in \mathcal{J}_N} \left(\exp(-\mu_j T) \cdot \langle \xi, h_j \rangle + \sum_{i \in \mathcal{I}_N} \lambda_i^{1/2} \cdot \widehat{Z}_{ij,N}(T) \right) \cdot h_j \tag{3.13}$$

as an approximation for $X(T)$.

3.2 Optimal Algorithms for Decoupled Systems of Equations

In this section we consider the stochastic evolution equation (3.1) with the particular noise $B(t) = I$ for every $t \in [0, T]$, where I is the identity operator on H. Thus, the Fourier expansion of the mild solution (3.9) at time point T reduces to

$$X(T) = \sum_{i \in \mathbb{N}^d} \Big(\exp(-\mu_i T) \cdot \langle \xi, h_i \rangle + \lambda_i^{1/2} \cdot Y_i(T) \Big) \cdot h_i. \tag{3.14}$$

Here $(Y_i(t))_{t \in [0,T]}$, with $i \in \mathbb{N}^d$, are independent Ornstein-Uhlenbeck processes, which are given by

$$Y_i(t) = \int_0^t \exp(-\mu_i(t-s))\, d\beta_i(s). \tag{3.15}$$

Due to Lemma C.0.1, the process (3.15) satisfies the scalar stochastic differential equation

$$\begin{aligned} dY_i(t) &= -\mu_i Y_i(t)\, dt + d\beta_i(t), \quad 0 < t \leq T, \\ Y_i(0) &= 0, \end{aligned} \tag{3.16}$$

for every $i \in \mathbb{N}^d$ and therefore $(Y_i)_{i \in \mathbb{N}^d}$ solves a system of independent homogeneous linear stochastic differential equations with constant coefficients.

In the following, we construct algorithms $\widehat{X}_N^*$, $\widehat{X}_N^{\#}$, $\widehat{X}_N^{\mathrm{equi}}$ and $\widehat{X}_N^{\mathrm{uni}}$, which are weakly asymptotically optimal in the respective classes defined in Section 3.1. All these algorithms are of the form

$$\widehat{X}_N(T) = \sum_{i \in \mathcal{I}_N} \Big(\exp(-\mu_i T) \cdot \langle \xi, h_i \rangle + \lambda_i^{1/2} \cdot \widehat{Y}_{i,N}(T) \Big) \cdot h_i \tag{3.17}$$

with $N \in \mathbb{N}$, a finite set $\mathcal{I}_N \subset \mathbb{N}^d$ and use the drift-implicit Euler-Maruyama scheme $\widehat{Y}_{i,N}$ as an approximation of $Y_i(T)$. For a given time discretization (3.11) with

$$\Delta_{k,i} = t_{k+1,i} - t_{k,i}$$

and

$$\Delta_{k,i}\beta_i = \beta_i(t_{k+1,i}) - \beta_i(t_{k,i})$$

for $i \in \mathbb{N}^d$, $n_i \in \mathbb{N}$ and $k = 0, \ldots, n_i - 1$, the drift-implicit Euler-Maruyama scheme for (3.16) is defined by

$$\begin{aligned} \widehat{Y}_{i,N}(t_{k+1,i}) &= \widehat{Y}_{i,N}(t_{k,i}) - \mu_i \widehat{Y}_{i,N}(t_{k+1,i}) \cdot \Delta_{k,i} + \Delta_{k,i}\beta_i, \\ \widehat{Y}_{i,N}(0) &= 0, \end{aligned} \tag{3.18}$$

for $k = 0, \ldots, n_i - 1$ and arbitrary $i \in \mathbb{N}^d$.

Now, we construct $\widehat{X}_N^*(T)$ with $N \in \mathbb{N}$ as follows. For the spatial discretization of W, and therewith also X, we select a ball using a radius with respect to the Euclidean norm. This radius depends on the cost and on the parameters d, γ and α. In particular, we differ between larger and smaller smoothness of the noise. The ball is defined by

$$\mathcal{I}_N^* = \begin{cases} \left\{ i \in \mathbb{N}^d \,|\, |i|_2 \le N^{1/d} \right\}, & \text{if } \gamma + \alpha \le 3d, \\ \left\{ i \in \mathbb{N}^d \,|\, |i|_2 \le N^{2/(\gamma+\alpha-d)} \right\}, & \text{if } \gamma + \alpha > 3d. \end{cases} \tag{3.19}$$

The number of evaluations of β_i with $i \in \mathcal{I}_N^*$, that we choose, additionally depends on the ratio between λ_i and μ_i taken to a power p. Here we put

$$n_i^* = \begin{cases} \left\lceil (\lambda_i/\mu_i)^p \cdot N^{(\gamma+\alpha)p/d} \right\rceil, & \text{if } \gamma + \alpha < 3d, \\ \left\lceil (\lambda_i/\mu_i)^p \cdot N/\ln(N) \right\rceil, & \text{if } \gamma + \alpha = 3d, \\ \left\lceil (\lambda_i/\mu_i)^p \cdot N \right\rceil, & \text{if } \gamma + \alpha > 3d, \end{cases} \tag{3.20}$$

$$\text{with } p \in \mathbb{R} \text{ satisfying } \begin{cases} \frac{\gamma+\alpha-d}{2(\gamma+\alpha)} < p < \frac{d}{\gamma+\alpha}, & \text{if } \gamma + \alpha < 3d, \\ p = \frac{1}{3}, & \text{if } \gamma + \alpha = 3d, \\ \frac{d}{\gamma+\alpha} < p < \frac{\gamma+\alpha-d}{2(\gamma+\alpha)}, & \text{if } \gamma + \alpha > 3d. \end{cases}$$

Furthermore, we choose the so-called regular time discretization $(t_{k,i}^*)_{k=0,\ldots,n_i^*, i \in \mathcal{I}_N^*}$, which is generated by the density $\psi_i(t) = \exp(-\mu_i/3 \cdot (T-t))$, $t \in [0,T]$, with $i \in \mathcal{I}_N^*$, i.e.

$$\int_0^{t_{k,i}^*} \exp(-\mu_i/3 \cdot (T-t))\, dt = \frac{k}{n_i^*} \cdot \int_0^T \exp(-\mu_i/3 \cdot (T-t))\, dt$$

for $k = 0, \ldots, n_i^*$ and $i \in \mathcal{I}_N^*$. Thus, these regular time nodes are quantiles of the density ψ_i. They are already used in [MGRW07], [MGRW08] and [W08] to obtain weakly asymptocally optimal algorithms for the equations considered in the respective contributions. By inserting this discretization in (3.18), we obtain for every $i \in \mathcal{I}_N^*$ an approximation $\widehat{Y}_{i,N}^*(T)$ for the solution $Y_i(T)$ of (3.16). Finally, we define

$$\widehat{X}_N^*(T) = \sum_{i \in \mathcal{I}_N^*} \Big(\exp(-\mu_i T) \cdot \langle \xi, h_i \rangle + \lambda_i^{1/2} \cdot \widehat{Y}_{i,N}^*(T) \Big) \cdot h_i. \tag{3.21}$$

For the construction of $\widehat{X}_N^{\#}(T)$ we define the ball

$$\mathcal{I}_N^{\#} = \left\{ i \in \mathbb{N}^d \,|\, |i|_2 \leq N^{2/(\gamma+\alpha+d)} \right\}$$

and the number of evaluations

$$n^{\#} = n_i^{\#} = \left\lceil N^{(\gamma+\alpha-d)/(\gamma+\alpha+d)} \right\rceil .$$

Because this number has to be constant for every $i \in \mathcal{I}_N^{\#}$ the ratio of λ_i and μ_i is irrelevant, now. As above, we choose the regularly generated time discretization, here given by the family of sequences $(t_{k,i}^{\#})_{k=0,\ldots,n^{\#}, i \in \mathcal{I}_N^{\#}}$, and use it in (3.18), to obtain $\widehat{Y}_{i,N}^{\#}(T)$. With this approximation, we define

$$\widehat{X}_N^{\#}(T) = \sum_{i \in \mathcal{I}_N^{\#}} \Big(\exp(-\mu_i T) \cdot \langle \xi, h_i \rangle + \lambda_i^{1/2} \cdot \widehat{Y}_{i,N}^{\#}(T) \Big) \cdot h_i. \tag{3.22}$$

Next, we construct $\widehat{X}_N^{\text{equi}}(T)$. For this purpose, define the space discretization ball and the numbers of evaluations by

$$\mathcal{I}_N^{\text{equi}} = \begin{cases} \left\{ i \in \mathbb{N}^d \,|\, |i|_2 \leq N^{1/(\alpha+d)} \right\}, & \text{if } \gamma - \alpha < 3d, \\ \left\{ i \in \mathbb{N}^d \,|\, |i|_2 \leq N^{2/(\gamma+\alpha-d)} \right\}, & \text{if } \gamma - \alpha \geq 3d, \end{cases} \tag{3.23}$$

and

$$n_i^{\text{equi}} = \begin{cases} \left\lceil (\lambda_i/\mu_i)^q \cdot N^{(\alpha+(\gamma+\alpha)q)/(\alpha+d)} \right\rceil, & \text{if } \gamma - \alpha < 3d, \\ \left\lceil (\lambda_i/\mu_i)^q \cdot N/\ln(N) \right\rceil, & \text{if } \gamma - \alpha = 3d, \\ \left\lceil (\lambda_i/\mu_i)^q \cdot N \right\rceil, & \text{if } \gamma - \alpha > 3d, \end{cases} \tag{3.24}$$

$$\text{with } q \in \mathbb{R} \text{ satisfying } \begin{cases} 0 < q < \frac{d}{\gamma+\alpha}, & \text{if } \gamma - \alpha < 3d, \\ q = \frac{d}{\gamma+\alpha}, & \text{if } \gamma - \alpha = 3d, \\ \frac{d}{\gamma+\alpha} < q < \frac{\gamma-\alpha-d}{2(\gamma+\alpha)}, & \text{if } \gamma - \alpha > 3d. \end{cases} \tag{3.25}$$

Here we use again the ratio of the eigenvalues λ_i and μ_j as for $\widehat{X}^*_N(T)$ with an adapted exponent q. This algorithm uses an equidistant time discretization of W. So, we choose time nodes $t^{\mathrm{equi}}_{k,i} = k/n^{\mathrm{equi}}_i \cdot T$, $k = 0, \dots, n^{\mathrm{equi}}_i$, for $i \in \mathcal{I}^{\mathrm{equi}}_N$ and apply them to (3.18) with $n_i = n^{\mathrm{equi}}_i$. Thus, we define

$$\widehat{X}^{\mathrm{equi}}_N(T) = \sum_{i \in \mathcal{I}^{\mathrm{equi}}_N} \Big(\exp(-\mu_i T) \cdot \langle \xi, h_i \rangle + \lambda_i^{1/2} \cdot \widehat{Y}^{\mathrm{equi}}_{i,N}(T) \Big) \cdot h_i. \tag{3.26}$$

At last, the construction of $\widehat{X}^{\mathrm{uni}}_N(T)$ is to do. Therefore we put

$$\mathcal{I}^{\mathrm{uni}}_N = \begin{cases} \left\{ i \in \mathbb{N}^d \mid |i|_2 \leq N^{1/(\alpha+d)} \right\}, & \text{if } \gamma - \alpha < d, \\ \left\{ i \in \mathbb{N}^d \mid |i|_2 \leq N^{2/(\gamma+\alpha+d)} \right\}, & \text{if } \gamma - \alpha \geq d, \end{cases} \tag{3.27}$$

and

$$n^{\mathrm{uni}} = n^{\mathrm{uni}}_i = \begin{cases} \left\lceil N^{\alpha/(\alpha+d)} \right\rceil, & \text{if } \gamma - \alpha < d, \\ \left\lceil N^{(\gamma+\alpha-d)/(\gamma+\alpha+d)} \right\rceil, & \text{if } \gamma - \alpha \geq d. \end{cases} \tag{3.28}$$

An uniform time discretization of the process W is chosen by selecting the time nodes $t^{\mathrm{uni}}_k = t^{\mathrm{uni}}_{k,i} = k/n^{\mathrm{uni}} \cdot T$, $k = 0, \dots, n^{\mathrm{uni}}$, for every $i \in \mathcal{I}^{\mathrm{uni}}_N$. By combining these nodes with (3.18), we receive $\widehat{Y}^{\mathrm{uni}}_{i,N}(T)$ with $n_i = n^{\mathrm{uni}}$ and we define

$$\widehat{X}^{\mathrm{uni}}_N(T) = \sum_{i \in \mathcal{I}^{\mathrm{uni}}_N} \Big(\exp(-\mu_i T) \cdot \langle \xi, h_i \rangle + \lambda_i^{1/2} \cdot \widehat{Y}^{\mathrm{uni}}_{i,N}(T) \Big) \cdot h_i. \tag{3.29}$$

Now, we state the following theorem about the asymptotic behaviour of the Nth minimal errors and that the constructed algorithms are weakly asymptotically optimal in their respective classes of approximations in the case that ξ is sufficiently smooth.

Theorem 3.2.1 *In the (ID) case,*

$$\mathrm{e}^*_N \asymp \begin{cases} N^{-(\alpha-1)/2}, & \text{if} \quad \alpha < 3, \\ N^{-1} \cdot (\ln N)^{3/2}, & \text{if} \quad \alpha = 3, \\ N^{-1}, & \text{if} \quad \alpha > 3, \end{cases} \tag{3.30}$$

$$\mathrm{e}^{\#}_N \asymp N^{-(\alpha-1)/(\alpha+1)}, \tag{3.31}$$

$$\mathrm{e}^{\mathrm{equi}}_N \asymp \mathrm{e}^{\mathrm{uni}}_N \asymp N^{-(\alpha-1)/(2(\alpha+1))} \tag{3.32}$$

and in the (TC) case,

$$\mathrm{e}_N^* \asymp \begin{cases} N^{-(\gamma+\alpha-d)/(2d)}, & \textit{if} \quad \gamma+\alpha < 3d, \\ N^{-1} \cdot (\ln N)^{3/2}, & \textit{if} \quad \gamma+\alpha = 3d, \\ N^{-1}, & \textit{if} \quad \gamma+\alpha > 3d, \end{cases} \tag{3.33}$$

$$\mathrm{e}_N^{\#} \asymp N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)}, \tag{3.34}$$

$$\mathrm{e}_N^{\mathrm{equi}} \asymp \begin{cases} N^{-(\gamma+\alpha-d)/(2(\alpha+d))}, & \textit{if} \quad \gamma-\alpha < 3d, \\ N^{-1} \cdot (\ln N)^{3/2}, & \textit{if} \quad \gamma-\alpha = 3d, \\ N^{-1}, & \textit{if} \quad \gamma-\alpha > 3d, \end{cases} \tag{3.35}$$

$$\mathrm{e}_N^{\mathrm{uni}} \asymp \begin{cases} N^{-(\gamma+\alpha-d)/(2(\alpha+d))}, & \textit{if} \quad \gamma-\alpha < d, \\ N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot (\ln N)^{1/2}, & \textit{if} \quad \gamma-\alpha = d, \\ N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)}, & \textit{if} \quad \gamma-\alpha > d. \end{cases} \tag{3.36}$$

Let furthermore $\langle \xi, h_i \rangle^2 \preceq \lambda_i$ *for every* $i \in \mathbb{N}^d$. *Then, in both cases,*

$$\mathrm{e}\left(\widehat{X}_N^*(T)\right) \asymp \mathrm{e}_N^*, \quad \mathrm{e}\left(\widehat{X}_N^{\#}(T)\right) \asymp \mathrm{e}_N^{\#}, \quad \mathrm{e}\left(\widehat{X}_N^{\mathrm{equi}}(T)\right) \asymp \mathrm{e}_N^{\mathrm{equi}}, \quad \mathrm{e}\left(\widehat{X}_N^{\mathrm{uni}}(T)\right) \asymp \mathrm{e}_N^{\mathrm{uni}}.$$

In the Figures 3.5 to 3.10 we illustrate and compare the rates of convergence given by the optimal algorithms in the different classes. Here we fix d and α and vary the smoothness parameter γ. We see that for smaller values of γ both classes using non-equidistant time discretizations are superior over the ones using equidistant time nodes. Note that the order of e_N^* even exceeds the one of $\mathrm{e}_N^{\#}$. We also find out that the minimal errors $\mathrm{e}_N^{\mathrm{equi}}$ and $\mathrm{e}_N^{\mathrm{uni}}$ are of the same quality in the case of little smoothness. In every algorithm class, increasing the smoothness leads to a larger order of the error except the limiting rate of 1 has already reached. In the classes $\mathfrak{X}_N^*$ and $\mathfrak{X}_N^{\mathrm{equi}}$ this gain is linear until it stops and stays at 1. In the class $\mathfrak{X}_N^{\mathrm{uni}}$ the order grows also linear, at first. But at a special point for γ, depending on α and d, the slope decreases and it is only heading asymptotically versus 1 together with the order in class $\mathfrak{X}_N^{\#}$. Now, the errors in these classes are of the same quality and the algorithm class $\mathfrak{X}_N^{\#}$ has become suboptimal with respect to the class $\mathfrak{X}_N^{\mathrm{equi}}$.

Refer to Section 2.4 for examples of stochastic evolution equations, which fulfil the requirements of Theorem 3.2.1. The results of the theorem generalize the results

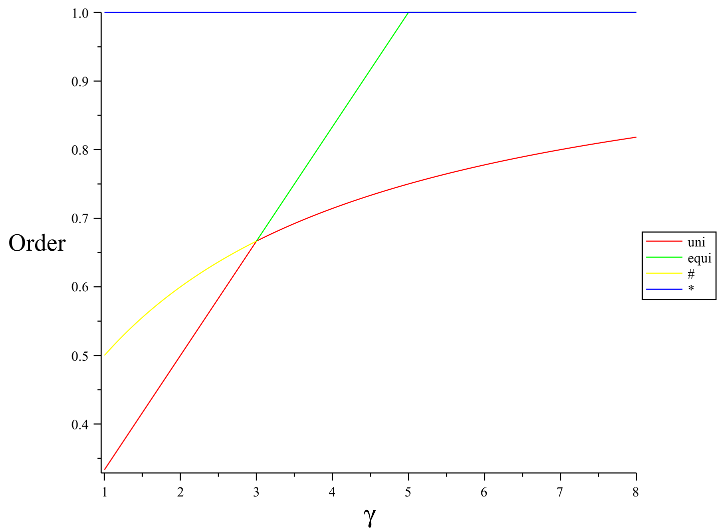

Figure 3.5: Optimal order of convergence in the case (TC), $d = 1$ and $\alpha = 2$

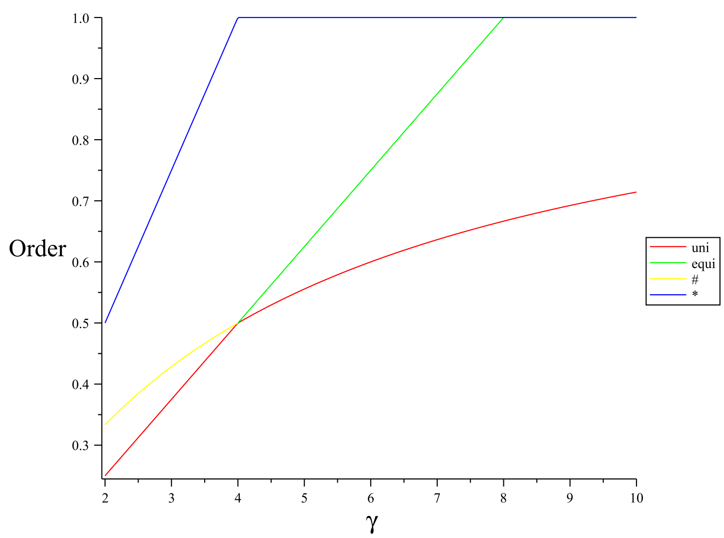

Figure 3.6: Optimal order of convergence in the case (TC), $d = 2$ and $\alpha = 2$

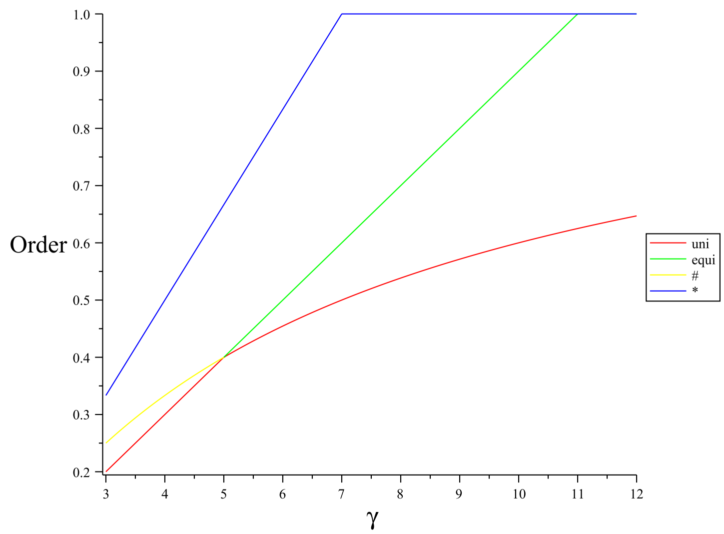

Figure 3.7: Optimal order of convergence in the case (TC), $d = 3$ and $\alpha = 2$

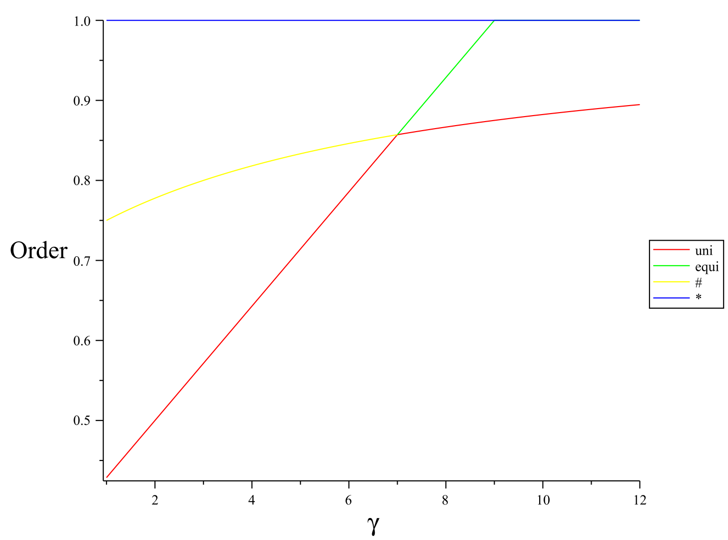

Figure 3.8: Optimal order of convergence in the case (TC), $d = 1$ and $\alpha = 6$

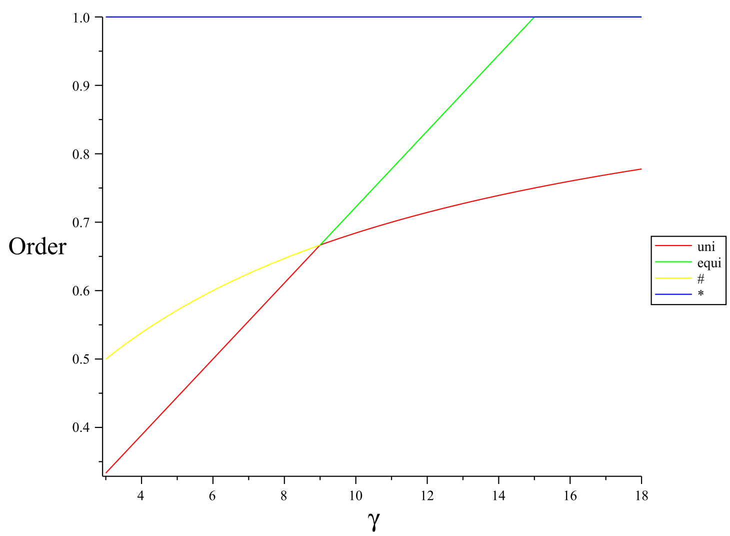

Figure 3.9: Optimal order of convergence in the case (TC), $d = 3$ and $\alpha = 6$

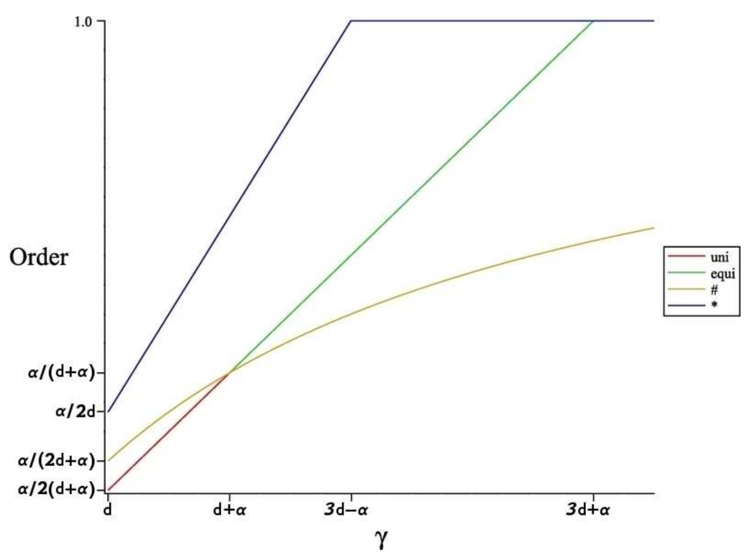

Figure 3.10: Optimal order of convergence in the case (TC), $d \in \mathbb{N}$ and $\alpha \geq 2$

given in [MGRW08] for a stochastic heat equation where the authors construct weakly asymptotically optimal algorithms in the classes $\mathfrak{X}_N^*$, $\mathfrak{X}_N^{\mathrm{equi}}$ and $\mathfrak{X}_N^{\mathrm{uni}}$, and provide the asymptotic behaviour of the respective minimal errors.

3.3 Algorithms for Coupled Systems of Equations

In this section we consider the stochastic evolution equation (3.1) with a state-independent noise satisfying Assumption 3.0.2. In the following we put

$$B_{ij}(t) = \langle B(t)h_i, h_j\rangle \tag{3.37}$$

for $t \in [0, T]$ and $i, j \in \mathbb{N}^d$. Thus, the Fourier expansion of the mild solution (3.9) at time point T is given by

$$X(T) = \sum_{j\in\mathbb{N}^d} \left(\exp(-\mu_j T) \cdot \langle \xi, h_j\rangle + \sum_{i\in\mathbb{N}^d} \lambda_i^{1/2} \cdot Z_{ij}(T) \right) \cdot h_j \tag{3.38}$$

with

$$Z_{ij}(t) = \int_0^t \exp(-\mu_j(t-s)) \cdot B_{ij}(s)\, d\beta_i(s) \tag{3.39}$$

for $t \in [0, T]$ and $i, j \in \mathbb{N}^d$. Since $B_{ij} \in C^1([0, T])$ for $i, j \in \mathbb{N}^d$ by assumption, we obtain

$$Z_{ij}(t) = B_{ij}(t)\beta_i(t) - \int_0^t \frac{\partial}{\partial s} \left(\exp(-\mu_j(t-s)) \cdot B_{ij}(s)\right) \beta_i(s)\, dt \tag{3.40}$$

for $t \in [0, T]$ by using the product formula for stochastic integration of Lemma C.0.1. Furthermore, the process (3.39) satisfies the scalar stochastic differential equation

$$\begin{aligned} dZ_{ij}(t) &= -\mu_j Z_{ij}(t)\, dt + B_{ij}(t)\, d\beta_i(t), \quad 0 < t \le T, \\ Z_{ij}(0) &= 0. \end{aligned} \tag{3.41}$$

The processes $(Z_{ij}(t))_{t\in[0,T]}$, with $i, j \in \mathbb{N}^d$, form a coupled system of Ornstein-Uhlenbeck processes in the case that the mapping B is independent of the time variable, i.e. $B(t) = B$ for every $t \in [0, T]$.

Now, we compare the quality of approximations using an uniform time discretization with the ones based upon non-uniform time discretizations. Specifically we consider the classes $\mathfrak{X}_N^{\mathrm{uni}}$ and $\mathfrak{X}_N^{\#}$, and provide algorithms $\widehat{X}_N^{\mathrm{uni}}(T)$ and $\widehat{X}_N^{\#}(T)$ that are weakly asymptotically optimal in the respective classes for a large number of combinations of d and the parameters α, β and γ introduced in the Assumptions 3.0.1 to 3.0.3.

At first, we construct $\widehat{X}_N^{\mathrm{uni}}(T)$. For this purpose, we consider the drift-implicit Euler-Maruyama scheme $\widehat{Z}_{ij,N}^{\mathrm{uni}}$, using uniform time nodes, to approximate the solution of (3.41). This means analog as in Section 3.2, for a given time discretization of W of the form (3.11) with

$$\Delta_{k,i} = t_{k+1,i} - t_{k,i}$$

and

$$\Delta_{k,i}\beta_i = \beta_i(t_{k+1,i}) - \beta_i(t_{k,i})$$

for $i \in \mathbb{N}^d$, $n_i \in \mathbb{N}$ and $k = 0, \dots, n_i - 1$, we define

$$\begin{aligned} \widehat{Z}_{ij,N}(t_{k+1,i}) &= \widehat{Z}_{ij,N}(t_{k,i}) - \mu_j \widehat{Z}_{ij,N}(t_{k+1,i}) \cdot \Delta_{k,i} + B_{ij}(t_{k,i}) \cdot \Delta_{k,i}\beta_i, \\ \widehat{Z}_{ij,N}(0) &= 0 \end{aligned} \tag{3.42}$$

for $k = 0, \dots, n_i - 1$ and arbitrary $i, j \in \mathbb{N}^d$. Here the approximation scheme should use an uniform time discretization, all the selected scalar Brownian motions are evaluated at. So, put $n = n_i$ and

$$t_k = t_{k,i} = k/n \cdot T, \quad k = 0, \dots, n,$$

for arbitrary $i \in \mathbb{N}^d$. We insert these nodes in (3.42) to obtain $\widehat{Z}_{ij,N}^{\mathrm{uni}}$.

Now, we provide the space discretization of the noise W and of the solution X used by $\widehat{X}_N^{\mathrm{uni}}(T)$. As in Section 3.2, we use for the spatial discretization of W a ball, which radius is expressed by the Euclidean norm. This radius here depends on d, γ, β and α. Thus, we set

$$\mathcal{I}_N^{\mathrm{uni}} = \left\{ i \in \mathbb{N}^d \,\middle|\, |i|_2 \le N^{P_{\mathcal{I}}} \right\} \tag{3.43}$$

with an exponent $P_{\mathcal{I}} > 0$ given below. In contrast, for the spatial disretization of X we use a so-called hyperbolic cross

$$\mathcal{J}_N^{\mathrm{uni}} = \left\{ j \in \mathbb{N}^d \,\middle|\, \prod_{\ell=1}^{d} j_\ell \le N^{P_{\mathcal{J}}} \right\}, \tag{3.44}$$

with an exponent $P_{\mathcal{J}} > 0$ stated later on. The size of the cross also depends on d, γ, β and α. Such a hyperbolic cross is already used in [MGR07a] to provide optimal methods for a stochastic heat equation with additive noise with respect to a global error criterion. We set

$$n^{\mathrm{uni}} = n_i^{\mathrm{uni}} = \lceil N^{P_n} \rceil \tag{3.45}$$

as the constant number of evaluations of β_i for every $i \in \mathcal{I}_N^{\mathrm{uni}}$ with a fixed exponent $P_n > 0$.

Next, we state the exponents $P_{\mathcal{I}}$, $P_{\mathcal{J}}$ and P_n used in (3.43), (3.44) and (3.45), which depend on d, γ, β and α. We define $\zeta = \min(\alpha, \beta)$ and $\eta = \min(\beta, \gamma)$ for notational convenience. In the case $d = 1$, put

$$P_{\mathcal{I}} = \begin{cases} \frac{\alpha+\eta-1}{\alpha(\gamma+\zeta)+\eta-1}, & \text{if } \eta - \alpha < 1, \\ \frac{2}{\gamma+\zeta+1}, & \text{if } \eta - \alpha \geq 1, \end{cases} \tag{3.46}$$

$$P_{\mathcal{J}} = \begin{cases} \frac{\gamma+\zeta-1}{\alpha(\gamma+\zeta)+\eta-1}, & \text{if } \eta - \alpha < 1, \\ \frac{\gamma+\zeta-1}{\alpha(\gamma+\zeta+1)}, & \text{if } \eta - \alpha = 1, \\ \frac{2(\gamma+\zeta-1)}{(\gamma+\zeta+1)(\alpha+\eta-1)}, & \text{if } \eta - \alpha > 1, \end{cases} \tag{3.47}$$

and

$$P_n = \begin{cases} \frac{\alpha(\gamma+\zeta)-\alpha}{\alpha(\gamma+\zeta)+\eta-1}, & \text{if } \eta - \alpha < 1, \\ \frac{\gamma+\zeta-1}{\gamma+\zeta+1}, & \text{if } \eta - \alpha \geq 1. \end{cases} \tag{3.48}$$

Note that we obtain

$$P_{\mathcal{I}} = \frac{\alpha - 1}{\alpha\zeta - 1}, \quad P_{\mathcal{J}} = \frac{\zeta - 1}{\alpha\zeta - 1} \quad \text{and} \quad P_n = \frac{\alpha(\zeta - 1)}{\alpha\zeta - 1}$$

in the (ID) case.

In the case $d \in \mathbb{N}\setminus\{1\}$, we only consider processes of higher smoothness where $\gamma \geq \beta \cdot d$ is satisfied because we only have results for those parameters. Then $\eta = \beta$ and we put

$$P_{\mathcal{I}} = \begin{cases} \frac{2((\beta-1)d+\alpha)}{2d((\beta-1)d+\alpha)+(\gamma+\zeta-d)((d+1)\alpha+d(d-1))}, & \text{if } \beta - \alpha < d, \\ \frac{2}{\gamma+\zeta+d}, & \text{if } \beta - \alpha \geq d, \end{cases} \tag{3.49}$$

$$P_{\mathcal{J}} = \begin{cases} \frac{2d(\gamma+\zeta-d)}{2d((\beta-1)d+\alpha)+(\gamma+\zeta-d)((d+1)\alpha+d(d-1))}, & \text{if } \beta - \alpha < d, \\ \frac{2d(\gamma+\zeta-d)}{(\gamma+\zeta+d)((\beta-1)d+\alpha)}, & \text{if } \beta - \alpha \geq d, \end{cases} \tag{3.50}$$

$$P_n = \begin{cases} \frac{(\gamma+\zeta-d)((d+1)\alpha+d(d-1))}{2d((\beta-1)d+\alpha)+(\gamma+\zeta-d)((d+1)\alpha+d(d-1))}, & \text{if } \beta-\alpha<d, \\ \frac{\gamma+\zeta-d}{\gamma+\zeta+d}, & \text{if } \beta-\alpha\geq d. \end{cases} \tag{3.51}$$

Finally, we define

$$\widehat{X}_N^{\mathrm{uni}}(T) = \sum_{j\in\mathcal{J}_N^{\mathrm{uni}}} \left(\exp(-\mu_j T)\cdot\langle \xi, h_j\rangle + \sum_{i\in\mathcal{I}_N^{\mathrm{uni}}} \lambda_i^{1/2}\cdot \widehat{Z}_{ij,N}^{\mathrm{uni}}(T) \right)\cdot h_j \tag{3.52}$$

as an approximation of $X(T)$.

Furthermore in this section, we define an approximation $\widehat{X}_N^{\#}(T)$. It uses a time discretization based on the regular chosen time nodes we just used in Section 3.2. As for the approximation $\widehat{X}_N^{\mathrm{uni}}(T)$, we set a ball of the form

$$\mathcal{I}_N^{\#} = \left\{ i\in\mathbb{N}^d \mid |i|_2 \leq N^{P_\mathcal{I}} \right\} \tag{3.53}$$

for the space discretization of W and a hyperbolic cross of the form

$$\mathcal{J}_N^{\#} = \left\{ j\in\mathbb{N}^d \;\middle|\; \prod_{\ell=1}^{d} j_\ell \leq N^{P_\mathcal{J}} \right\} \tag{3.54}$$

for the space discretization of X, with fixed $P_\mathcal{I}$, $P_\mathcal{J} > 0$ given below. For the construction of a time discretization of W with the help of regular time nodes, we choose an evaluation number $\nu_j\in\mathbb{N}$, stated explicitly below, for every $j\in\mathcal{J}_N^{\#}$. With these numbers, we consider the time nodes

$$0 < s_{1,j} < \ldots < s_{\nu_j,j} = T$$

for $j\in\mathcal{J}_N^{\#}$, which are regularly generated by the density $\psi_j(t) = \exp(-\mu_j/3\cdot(T-t))$, $t\in[0,T]$, i.e.

$$\int_0^{s_{k,j}} \exp(-\mu_j/3\cdot(T-t))\,dt = \frac{k}{\nu_j}\int_0^T \exp(-\mu_j/3\cdot(T-t))\,dt$$

for $j\in\mathbb{N}^d$, $\nu_j\in\mathbb{N}$ and $k=0,\ldots,\nu_j$. We take these time nodes to estimate $Z_{ij}(T)$ in (3.38) by defining

$$\widehat{Z}_{ij,N}^{\#}(T) = \sum_{k=0}^{\nu_j-1} B_{ij}(s_{k,j})\cdot(\beta_i(s_{k+1,j})-\beta_i(s_{k,j}))\prod_{\ell=k}^{\nu_j-1}(1+\mu_j\cdot(s_{\ell+1,j}-s_{\ell,j}))^{-1} \tag{3.55}$$

for $i \in \mathcal{I}_N^{\#}$ and $j \in \mathcal{J}_N^{\#}$ having the drift-implizit Euler-Maruyama scheme (3.42) in mind. By this construction of a time discretization, note that the ith scalar Brownian motion β_i, with $i \in \mathcal{I}_N^{\#}$, is evaluated at the time nodes

$$0 < t_1 \leq \ldots \leq t_n = T$$

with $n = n_i$ and

$$\{t_1 \ldots, t_n\} = \bigcup_{j \in \mathcal{J}_N^{\#}} \left\{s_{1,j}, \ldots, s_{\nu_j,j}\right\}.$$

Therefore, every β_i with $i \in \mathcal{I}_N^{\#}$ uses all the time nodes generated by the densities ψ_j with $j \in \mathcal{J}_N^{\#}$.

Now, we give the exponents $P_{\mathcal{I}}$ and $P_{\mathcal{J}}$ in (3.53) and (3.54) as well as the numbers ν_j, with $j \in \mathcal{J}_N^{\#}$, which all depend on d, γ, β and α. Remember that $\zeta = \min(\alpha, \beta)$ and $\eta = \min(\beta, \gamma)$.

In cases of higher smoothness such that $\gamma \geq \beta \cdot d$, we have $\eta = \beta$ and put for the exponent of the ball

$$P_{\mathcal{I}} = \begin{cases} \frac{2((\beta-1)d+\alpha)}{2d((\beta-1)d+\alpha)+(\gamma+\zeta-d)((3d-\alpha-1)d+\alpha)}, & \text{if } \beta+\alpha < 3d, \\ \frac{2}{\gamma+\zeta+d}, & \text{if } \beta+\alpha \geq 3d, \end{cases} \tag{3.56}$$

and for the exponent of the hyperbolic cross

$$P_{\mathcal{J}} = \begin{cases} \frac{2d(\gamma+\zeta-d)}{2d((\beta-1)d+\alpha)+(\gamma+\zeta-d)((3d-\alpha-1)d+\alpha)}, & \text{if } \beta+\alpha < 3d, \\ \frac{2d(\gamma+\zeta-d)}{(\gamma+\zeta+d)((\beta-1)d+\alpha)}, & \text{if } \beta+\alpha \geq 3d. \end{cases} \tag{3.57}$$

As in Section 3.2 we define the number of the considered regular time nodes with respect to the ratio of the respective eigenvalues of the operators Q and A. So, we set

$$\nu_j = \begin{cases} \lceil (\lambda_j/\mu_j)^{P_\mu} \cdot N^{P_\nu} \rceil, & \text{if } \beta+\alpha \neq 3d, \\ \lceil (\lambda_j/\mu_j)^{P_\mu} \cdot N^{P_\nu} / \ln N \rceil, & \text{if } \beta+\alpha = 3d, \end{cases} \tag{3.58}$$

$$\text{with } P_\mu \text{ satisfying } \begin{cases} \frac{\beta+\alpha-d}{2(\gamma+\alpha)} < P_\mu < \frac{d}{\gamma+\alpha}, & \text{if } \beta+\alpha < 3d, \\ P_\mu = \frac{d}{\gamma+\alpha}, & \text{if } \beta+\alpha = 3d, \\ \frac{d}{\gamma+\alpha} < P_\mu < \frac{\beta+\alpha-d}{2(\gamma+\alpha)}, & \text{if } \beta+\alpha > 3d, \end{cases} \tag{3.59}$$

$$\text{and } P_\nu = \begin{cases} \frac{(\gamma+\zeta-d)((d-\alpha-1)d+\alpha+2d(\gamma+\alpha)P_\mu)}{2d((\beta-1)d+\alpha)+(\gamma+\zeta-d)((3d-\alpha-1)d+\alpha)}, & \text{if } \beta+\alpha < 3d, \\ \frac{\gamma+\zeta-d}{\gamma+\zeta+d}, & \text{if } \beta+\alpha \geq 3d. \end{cases} \tag{3.60}$$

Otherwise for lower smoothness, where $\gamma < \beta \cdot d$, we express the exponents by a case distinction for the parameters γ and β at once. Therefore we use η and put

$$P_{\mathcal{I}} = \begin{cases} \frac{2(\gamma+\alpha-d)}{2d(\gamma+\alpha-d)+(\gamma+\zeta-d)((3d-\alpha-1)d+\alpha+\gamma-\eta d)}, & \text{if } \eta+\alpha < 3d, \\ \frac{2}{\gamma+\zeta+d}, & \text{if } \eta+\alpha \geq 3d, \end{cases} \tag{3.61}$$

for the ball radius,

$$P_{\mathcal{J}} = \begin{cases} \frac{2d(\gamma+\zeta-d)}{2d(\gamma+\alpha-d)+(\gamma+\zeta-d)((3d-\alpha-1)d+\alpha+\gamma-\eta d)}, & \text{if } \eta+\alpha < 3d, \\ \frac{2d(\gamma+\zeta-d)}{(\gamma+\zeta+d)(\gamma+\alpha-d)}, & \text{if } \eta+\alpha \geq 3d, \end{cases} \tag{3.62}$$

for the cross size and

$$\nu_j = \begin{cases} \lceil (\lambda_j/\mu_j)^{P_\mu} \cdot N^{P_\nu} \rceil, & \text{if } \eta+\alpha \neq 3d, \\ \lceil (\lambda_j/\mu_j)^{P_\mu} \cdot N^{P_\nu} / \ln N \rceil, & \text{if } \eta+\alpha = 3d, \end{cases} \tag{3.63}$$

$$\text{with } P_\mu \text{ satisfying } \begin{cases} \frac{\eta+\alpha-d}{2(\gamma+\alpha)} < P_\mu < \frac{d}{\gamma+\alpha}, & \text{if } \eta+\alpha < 3d, \\ P_\mu = \frac{d}{\gamma+\alpha}, & \text{if } \eta+\alpha = 3d, \\ \frac{d}{\gamma+\alpha} < P_\mu < \frac{\eta+\alpha-d}{2(\gamma+\alpha)}, & \text{if } \eta+\alpha > 3d, \end{cases} \tag{3.64}$$

$$\text{and } P_\nu = \begin{cases} \frac{(\gamma+\zeta-d)((d-\alpha-1)d+\alpha+\gamma-\eta d+2d(\gamma+\alpha)P_\mu)}{2d(\gamma+\alpha-d)+(\gamma+\zeta-d)((3d-\alpha-1)d+\alpha+\gamma-\eta d)}, & \text{if } \eta+\alpha < 3d, \\ \frac{\gamma+\zeta-d}{\gamma+\zeta+d}, & \text{if } \eta+\alpha \geq 3d, \end{cases} \tag{3.65}$$

as the number of the regular time nodes.

In the important special case $d = 1$, the above settings reduces to

$$P_{\mathcal{I}} = \begin{cases} \frac{\eta+\alpha-1}{\eta+\alpha+\gamma+\zeta-2}, & \text{if } \eta+\alpha < 3, \\ \frac{2}{\gamma+\zeta+1}, & \text{if } \eta+\alpha \geq 3, \end{cases} \tag{3.66}$$

$$P_{\mathcal{J}} = \begin{cases} \frac{\gamma+\zeta-1}{\eta+\alpha+\gamma+\zeta-2}, & \text{if } \eta+\alpha < 3, \\ \frac{2(\gamma+\zeta-1)}{(\gamma+\zeta+1)(\eta+\alpha-1)}, & \text{if } \eta+\alpha \geq 3, \end{cases} \tag{3.67}$$

$$P_\nu = \begin{cases} \frac{(\gamma+\zeta-1)(\gamma+\alpha)P_\mu}{\eta+\alpha+\gamma+\zeta-2}, & \text{if } \eta+\alpha < 3, \\ \frac{\gamma+\zeta-1}{\gamma+\zeta+1}, & \text{if } \eta+\alpha \geq 3, \end{cases} \tag{3.68}$$

$$\text{and } P_\mu \text{ satisfying } \begin{cases} \frac{\eta+\alpha-1}{2(\gamma+\alpha)} < P_\mu < \frac{1}{\gamma+\alpha}, & \text{if } \eta+\alpha < 3, \\ P_\mu = \frac{1}{\gamma+\alpha}, & \text{if } \eta+\alpha = 3, \\ \frac{1}{\gamma+\alpha} < P_\mu < \frac{\eta+\alpha-1}{2(\gamma+\alpha)}, & \text{if } \eta+\alpha > 3. \end{cases} \tag{3.69}$$

In the (ID) case, we obtain

$$P_{\mathcal{I}} = \begin{cases} \frac{\alpha-1}{\alpha+\zeta-2}, & \text{if } \alpha < 3, \\ \frac{2}{\zeta+1}, & \text{if } \alpha \geq 3, \end{cases} \tag{3.70}$$

$$P_{\mathcal{J}} = \begin{cases} \frac{\zeta-1}{\alpha+\zeta-2}, & \text{if } \alpha < 3, \\ \frac{2(\zeta-1)}{(\zeta+1)(\alpha-1)}, & \text{if } \alpha \geq 3, \end{cases} \tag{3.71}$$

and

$$\nu_j = \begin{cases} \lceil (1/\mu_j)^{P_\mu} \cdot N^{P_\nu} \rceil, & \text{if } \alpha \neq 3, \\ \lceil (1/\mu_j)^{P_\mu} \cdot N^{P_\nu} / \ln N \rceil, & \text{if } \alpha = 3, \end{cases} \tag{3.72}$$

$$\text{with } P_\mu \text{ satisfying } \begin{cases} \frac{\alpha-1}{2\alpha} < P_\mu < \frac{1}{\alpha}, & \text{if } \alpha < 3, \\ P_\mu = \frac{1}{\alpha}, & \text{if } \alpha = 3, \\ \frac{1}{\alpha} < P_\mu < \frac{\alpha-1}{2\alpha}, & \text{if } \alpha > 3, \end{cases} \tag{3.73}$$

and

$$P_\nu = \begin{cases} \frac{\alpha(\zeta-1)P_\mu}{\alpha+\zeta-2}, & \text{if } \alpha < 3, \\ \frac{\zeta-1}{\zeta+1}, & \text{if } \alpha \geq 3. \end{cases} \tag{3.74}$$

Finally, we define

$$\widehat{X}_N^{\#}(T) = \sum_{j \in \mathcal{J}_N^{\#}} \left(\exp(-\mu_j T) \cdot \langle \xi, h_j \rangle + \sum_{i \in \mathcal{I}_N^{\#}} \lambda_i^{1/2} \cdot \widehat{Z}_{ij,N}^{\#}(T) \right) \cdot h_j \tag{3.75}$$

as approximation of $X(T)$ based on regular time discretizations.

Now, we state two theorems about the asymptotic behaviour of the Nth minimal errors. The first one covers the case $d = 1$ and gives in particular some combinations of the parameters α, β and γ for which we obtain weakly asymptotic optimality for the constructed algorithms.

Theorem 3.3.1 *Let $d = 1$ and suppose that $B_{ij} : [0, T] \to \mathbb{R}$ is constant, i.e.*

$$B_{ij} = B_{ij}(t), \quad t \in [0, T],$$

for every $i, j \in \mathbb{N}$. Then it holds

$$\mathrm{e}_N^{\#} \asymp N^{-(\gamma+\alpha-1)/(\gamma+\alpha+1)},$$

in the case that

$$\gamma + \alpha > 3 \quad \textit{and} \quad \max(\alpha, \gamma) \le \beta \tag{3.76}$$

or in the case that

$$\beta + \alpha > 3 \quad \textit{and} \quad \alpha \le \beta \le \gamma. \tag{3.77}$$

Also it holds

$$\mathrm{e}_N^{\mathrm{uni}} \asymp \begin{cases} N^{-(\gamma+\alpha-1)/(2(\alpha+1))}, & \textit{if } \gamma - \alpha < 1 \textit{ and } \max(\alpha, \gamma) \le \beta, \\ N^{-(\gamma+\alpha-1)/(\gamma+\alpha+1)} \cdot (\ln N)^{1/2}, & \textit{if } \gamma - \alpha = 1 \textit{ and } \alpha < \gamma \le \beta, \\ N^{-(\gamma+\alpha-1)/(\gamma+\alpha+1)}, & \textit{if } \min(\beta, \gamma) - \alpha > 1. \end{cases} \tag{3.78}$$

Additionally, suppose that

$$\langle \xi, h_j \rangle^2 \preceq \begin{cases} j^{-\gamma}, & \textit{if} \quad \gamma \le \beta, \\ j^{-\beta}, & \textit{if} \quad \gamma > \beta, \end{cases} \tag{3.79}$$

for every $j \in \mathbb{N}$. Then

$$\mathrm{e}\left(\widehat{X}_N^{\#}(T)\right) \asymp \mathrm{e}_N^{\#},$$

if the parameters satisfy (3.76) or (3.77), as well as

$$\mathrm{e}\left(\widehat{X}_N^{\mathrm{uni}}(T)\right) \asymp \mathrm{e}_N^{\mathrm{uni}},$$

if the parameters satisfy the respective conditions in (3.78).

In the (ID) case, the statements in Theorem 3.3.1 about the Nth minimal errors reduce to

$$\mathrm{e}_N^{\#} \asymp N^{-(\alpha-1)/(\alpha+1)},$$

if $3 < \alpha \le \beta$, and

$$\mathrm{e}_N^{\mathrm{uni}} \asymp N^{-(\alpha-1)/(2(\alpha+1))},$$

if $\alpha \le \beta$. Thus, $\mathrm{e}_N^{\mathrm{uni}} \asymp N^{-1/6}$ for the important specific values $\alpha = \beta = 2$. For this setting, we have not shown optimality in the class $\mathfrak{X}_N^{\#}$ but $\mathrm{e}(\widehat{X}_N^{\#}(T)) \preceq N^{-1/4}$ by Proposition 3.4.4 in Section 3.4, which we use to prove the theorem. That means that the convergence order of the upper bound of $\mathrm{e}(\widehat{X}_N^{\#}(T))$ exceeds the convergence order of the derived lower bound of $\mathrm{e}_N^{\mathrm{uni}}$. Therefore, we have here a superiority of $\widehat{X}_N^{\#}(T)$ over all algorithms $\widehat{X}(T) \in \mathfrak{X}_N^{\mathrm{uni}}$. By comparing the Propositions 3.4.4 and 3.4.7 in Section 3.4, we see that this superiority also occurs for further combinations of the parameters α and β. In detail, we state that the algorithm $\widehat{X}_N^{\#}(T)$ is superior over all algorithms $\widehat{X}(T) \in \mathfrak{X}_N^{\mathrm{uni}}$ in the (ID) case if one of the following conditions is fulfilled.

- $\alpha \le \beta$,
- $\frac{2\alpha-1}{\alpha} < \beta < \alpha < 3$,
- $\frac{5}{3} + \epsilon < \beta < \alpha = 3$ for an arbitrary small $\epsilon > 0$,
- $\frac{\alpha-1}{\alpha+1} < \frac{2(\beta-1)}{\beta+1}$, $\beta < \alpha$ and $\alpha > 3$.

For the third condition, we used that in Proposition 3.4.4 the term $\ln N$ can be estimated by N^{ϵ} for an arbitrary small $\epsilon > 0$.

In the (TC) case, if we set $\alpha = \beta = 2$, we get optimality in the class $\mathfrak{X}_N^{\#}$ for every $\gamma > 1$ and in the class $\mathfrak{X}_N^{\mathrm{uni}}$ for $1 < \gamma \le 2$. A superiority of $\widehat{X}_N^{\#}(T)$ over all algorithms $\widehat{X}(T) \in \mathfrak{X}_N^{\mathrm{uni}}$ occurs for $1 < \gamma < 3$. In the Figures 3.11 to 3.16, we illustrate and compare the convergence orders of the lower and upper error bounds, we obtain in the Propositions 3.4.4, 3.4.5 and 3.4.7 in Section 3.4, depending on the smoothness parameter γ for $d = 1$ and different fixed values α and β. We conclude that larger values of γ lead to a higher convergence order for every error bound. The derived orders with respect to the class $\mathfrak{X}_N^{\#}$ are heading asymptotically versus the limiting value 1. The orders for the class $\mathfrak{X}_N^{\mathrm{uni}}$ grow linearly at first and then switch to a strictly concave

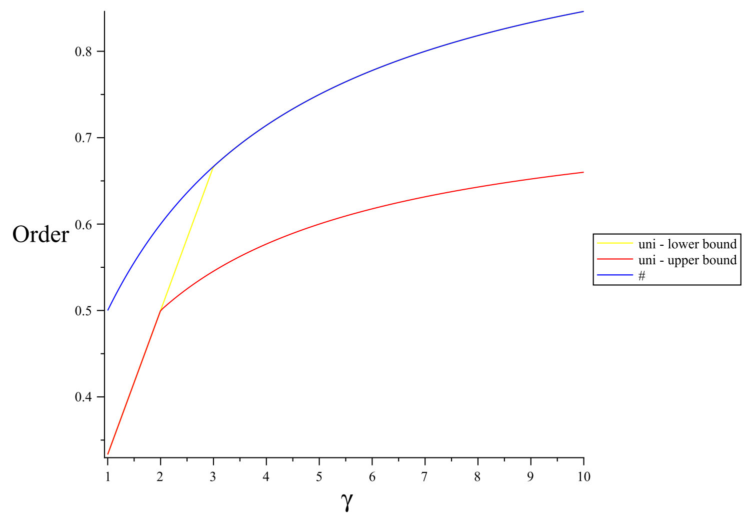

Figure 3.11: Order of convergence in the case (TC), $d = 1$ and $\alpha = \beta = 2$

increase. However, the asymptote for its upper error bound is a value smaller than 1, depending on α and β, in case of non-optimality.

If $\alpha > \beta$ we have not shown optimality for the constructed algorithms for any γ, only superiority of $\widehat{X}_N^{\#}(T)$ for smaller values of γ. But by increasing β also the lower and upper error bounds get closer in both algorithm classes, see Figures 3.13 and 3.14. At last for $\alpha = \beta$, optimality comes up for $\widehat{X}_N^{\#}(T)$ and also for $\widehat{X}_N^{\mathrm{uni}}(T)$ in the case of smaller smoothness, which we see in Figures 3.11 and 3.15. If even $\beta \geq \alpha + 1$ we gain optimality in both algorithm classes for any γ, compare Figures 3.12 and 3.16. Here for higher smoothness both algorithms are of the same quality and the non-uniform time discetization using the same evaluation number for every scalar component of W cannot bring more benefit.

For completion, we give all the combinations of the parameters α, β and γ for which the class $\mathfrak{X}_N^{\mathrm{uni}}$ is suboptimal with respect to the approximation $\widehat{X}_N^{\#}(T)$. The superiority of $\widehat{X}_N^{\#}(T)$ may occur for the parameters α and γ satisfying $\gamma - \alpha < 1$. If in addition $\eta + \alpha < 3$, we obtain superiority in one of the following settings.

- $\max(\alpha, \gamma) \leq \beta$,

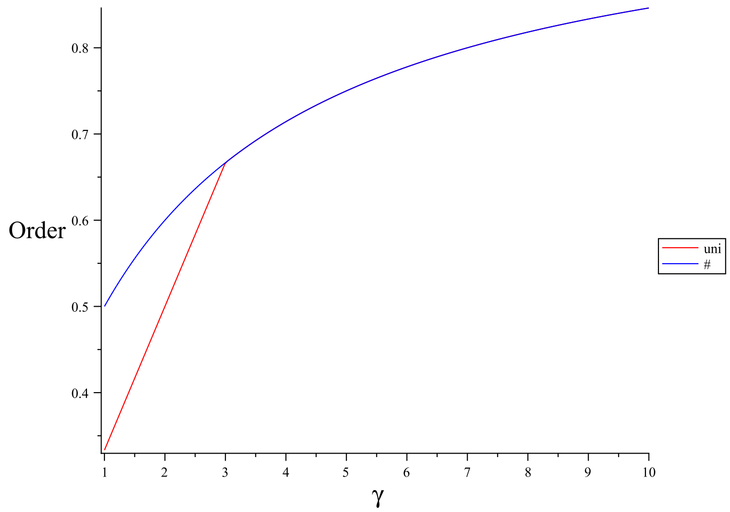

Figure 3.12: Optimal order of convergence in the case (TC), $d = 1$, $\alpha = 2$ and $\beta \geq 3$

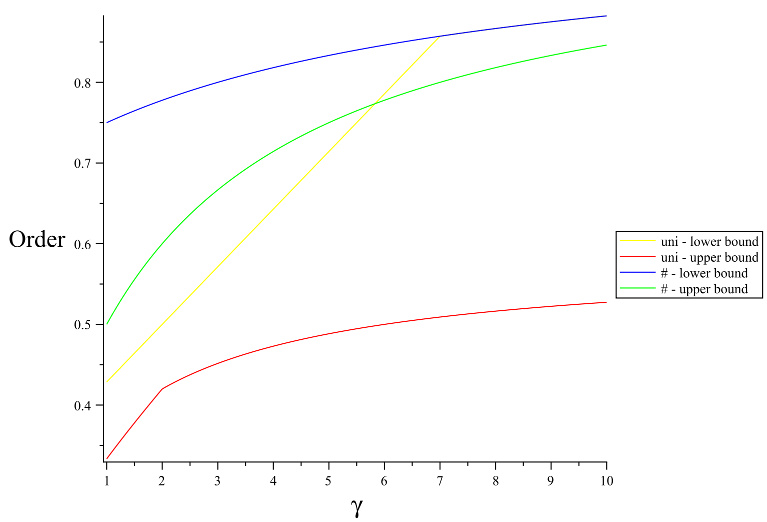

Figure 3.13: Order of convergence in the case (TC), $d = 1$, $\alpha = 6$ and $\beta = 2$

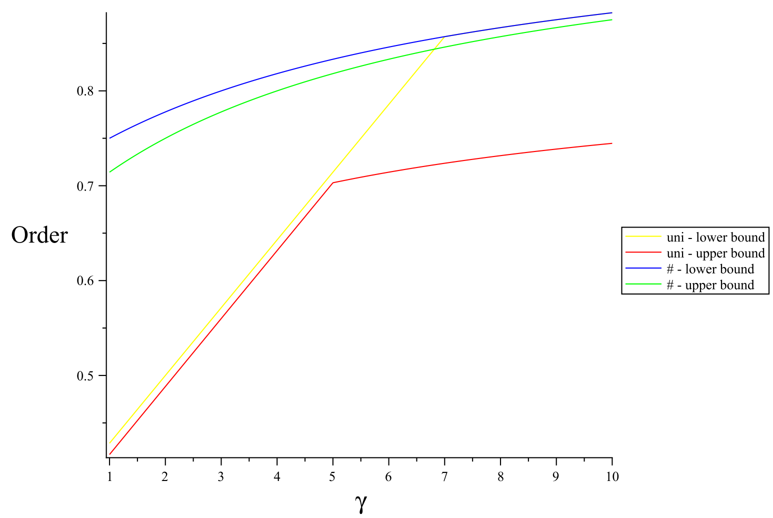

Figure 3.14: Order of convergence in the case (TC), $d = 1$, $\alpha = 6$ and $\beta = 5$

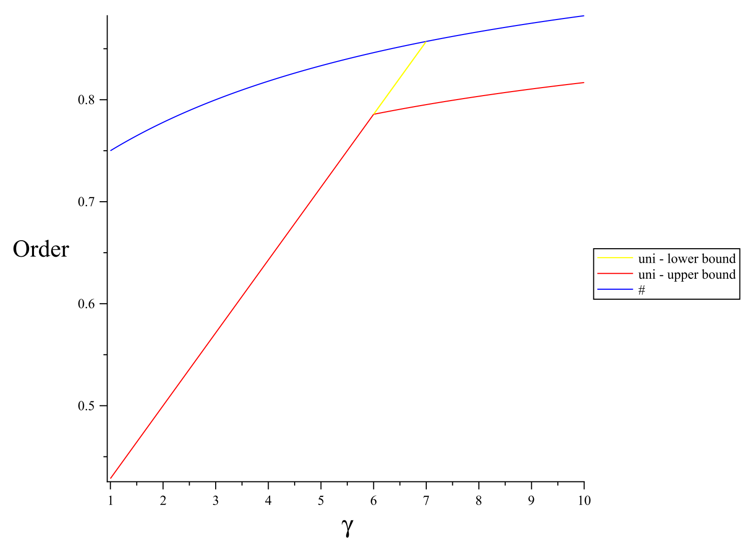

Figure 3.15: Order of convergence in the case (TC), $d = 1$, $\alpha = 6$ and $\beta = 6$

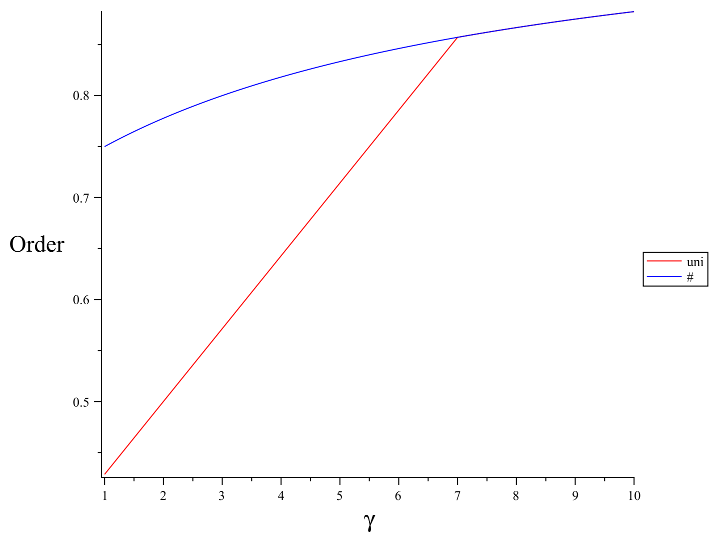

Figure 3.16: Optimal order of convergence in the case (TC), $d = 1$, $\alpha = 6$ and $\beta \geq 7$

- $\gamma \leq \beta \leq \alpha$,
- $\alpha \leq \beta \leq \gamma$ and $\alpha(\alpha + \beta - 2) > \gamma - 1$,
- $\beta \leq \min(\alpha, \gamma)$ and $(\gamma + \beta - 1)(\beta + \alpha - 1)(\alpha + 1) > (\gamma + \alpha - 1)(\gamma + \alpha + 2\beta - 2)$.

On the other hand, if in addition $\eta + \alpha > 3$, we have superiority in case that one of the following terms holds.

- $\max(\alpha, \gamma) \leq \beta$,
- $\alpha \leq \beta \leq \gamma$,
- $\gamma \leq \beta \leq \alpha$ and $2(\alpha + 1)(\gamma + \beta - 1) > (\gamma + \alpha - 1)(\gamma + \beta + 1)$,
- $\beta \leq \min(\alpha, \gamma)$ and $2(\alpha + 1)(\gamma + \beta - 1) > (\gamma + \alpha - 1)(\gamma + \beta + 1)$.

Finally, in the additional limiting case $\eta + \alpha = 3$, we get superiority for one of the following cases.

- $\max(\alpha, \gamma) \leq \beta$ and $\alpha > 1 + \epsilon$ for an arbitrary small $\epsilon > 0$,
- $\alpha \leq \beta \leq \gamma$ and $\gamma - \alpha < 1 - \epsilon$ for an arbitrary small $\epsilon > 0$,
- $\gamma \leq \beta \leq \alpha$ and $\alpha(2 + \beta - \alpha) > 2 + \epsilon$ for an arbitrary small $\epsilon > 0$,
- $\beta \leq \min(\alpha, \gamma)$.

Now, we state the second theorem, covering the case $d \in \mathbb{N} \setminus \{1\}$. Here we get no asymptotic optimality for the Nth minimal errors by combining the Propositions 3.4.4, 3.4.6 and 3.4.7 in Section 3.4, used for its proof, because of the logarithmic terms in (3.119), (3.120) and (3.141). Nevertheless, disregarding the logarithmic factor, we obtain weakly asymptotically optimality for some choices of the parameters d, α, β and γ.

Theorem 3.3.2 *Let $d \in \mathbb{N} \setminus \{1\}$ and suppose that $B_{ij} : [0, T] \to \mathbb{R}$ is constant, i.e.*

$$B_{ij} = B_{ij}(t), \quad t \in [0, T],$$

for every $i, j \in \mathbb{N}^d$. Then it holds

$$\mathrm{e}_N^{\#} \preceq N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot (\ln N)^{(d-1)/2} \tag{3.80}$$

in the case that

$$\beta + \alpha > 3d \;\; \textit{and} \;\; \alpha \leq \beta \leq \gamma$$

or in the case that

$$\gamma + \alpha > 3d \;\; \textit{and} \;\; \max(\alpha, \gamma) \leq \beta.$$

Also it holds

$$\mathrm{e}_N^{\mathrm{uni}} \preceq N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot (\ln N)^{(d-1)/2} \tag{3.81}$$

in the case that

$$\alpha \leq d, \;\; \gamma \geq \beta \cdot d \;\; \textit{and} \;\; \beta - \alpha > d.$$

All the given upper bounds of the respective Nth minimal errors are weakly asymptotically optimal under the respective stated conditions, disregarding the logarithmic factor.

Additionally, suppose that

$$\langle \xi, h_j \rangle^2 \preceq \begin{cases} \lambda_j + \prod_{\ell=1}^{d} j_\ell^{-\gamma/d}, & \text{if } \gamma < \beta \cdot d, \\ \prod_{\ell=1}^{d} j_\ell^{-\beta}, & \text{if } \gamma \geq \beta \cdot d, \end{cases} \tag{3.82}$$

for every $j \in \mathbb{N}^d$. Then the stated upper error bounds in (3.80) and (3.81) are respectively achieved by the corresponding algorithms $\widehat{X}_N^{\#}(T)$ and $\widehat{X}_N^{\mathrm{uni}}(T)$.

For the setting $d = \alpha = \beta = 2$, we illustrate in Figure 3.17 the lower and upper convergence orders, derived in the Propositions 3.4.4, 3.4.6 and 3.4.7 in Section 3.4, for a varying γ, disregarding the logarithmic term. We see again that the increase of the value γ leads to an improvement of every error bound. In fact, for $\gamma \geq 4$ the derived convergence orders of the lower bounds of the Nth minimal error in both algorithm classes coincide as well as those of the upper error bounds and move asymptotically towards 1 respectively towards 1/2. That means $\widehat{X}_N^{\mathrm{uni}}(T)$ and $\widehat{X}_N^{\#}(T)$ are of the same quality in this region. For $\gamma < 4$ we do not have results for an upper error bound in the class with uniform time discretization, whereas its lower bound grows linearly. We have not shown optimality for the constructed algorithms in this setting at all. However, just as in the case $d = 1$, increasing the decay parameter β leads to an approach of the upper and lower error bounds and finally to optimality in both classes, up to the logarithmic factor. Here for $\beta \geq 4$ the constructed algorithms in the case $\gamma \geq 4$ are optimal, see Figure 3.18. More general, if $\beta \geq d + \alpha$ we obtain optimality for both constructed algorithms with the same order of convergence in regions of higher smoothness. That means, we cannot benefit from the non-uniform time discretization in the class $\mathfrak{X}_N^{\#}$ to provide a superior approximation with respect to the class $\mathfrak{X}_N^{\mathrm{uni}}$. In Figures 3.20 to 3.22, we see the changing of the convergence orders for $d = \alpha = 6$ and the different values $\beta = 2$, $\beta = 6$ and $\beta = 12$. In the latter setting optimality is achieved.

Analog to $d = 1$, we obtain for $d \in \mathbb{N} \setminus \{1\}$ superiority of $\widehat{X}_N^{\#}(T)$ over all algorithms $\widehat{X}(T) \in \mathfrak{X}_N^{\mathrm{uni}}$ if d, α, β and γ are chosen conveniently, see Figure 3.19. To complete our studies we give a formal overview on those parameters. For this superiority, it always holds $\gamma - \alpha < d$. In the case that in addition $\eta + \alpha > 3d$, we need furthermore one of the following conditions for an arbitrary small $\epsilon > 0$.

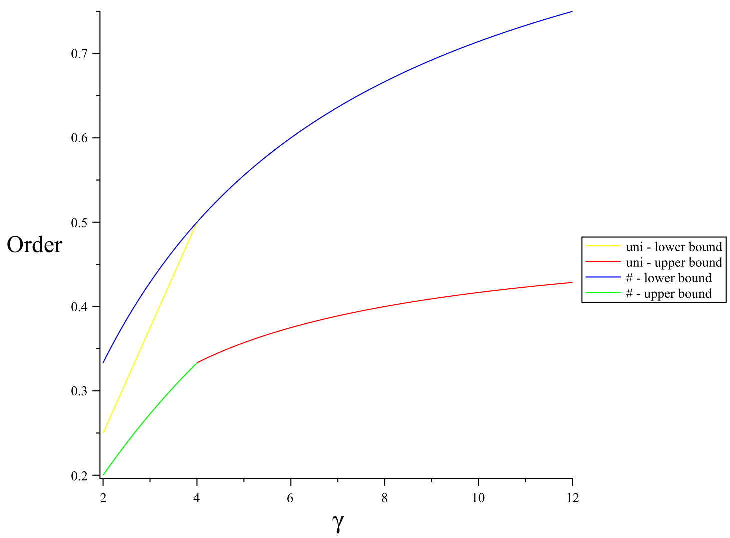

Figure 3.17: Order of convergence in the case (TC), $d = 2$ and $\alpha = \beta = 2$

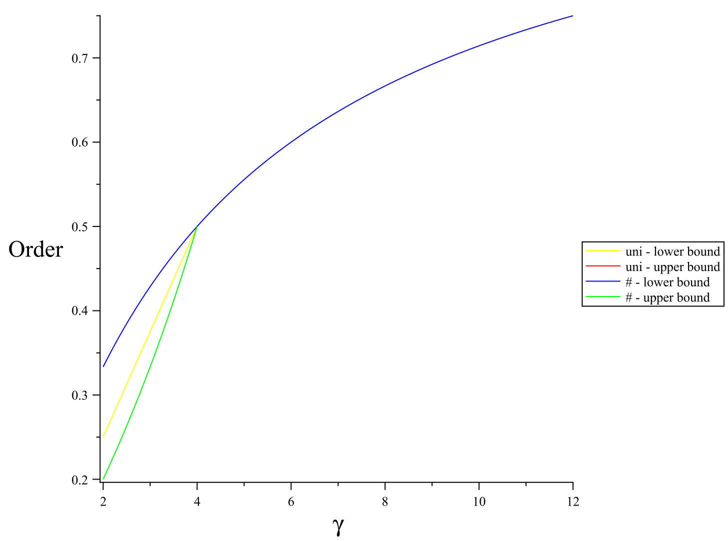

Figure 3.18: Order of convergence in the case (TC), $d = 2$, $\alpha = 2$ and $\beta = 4$

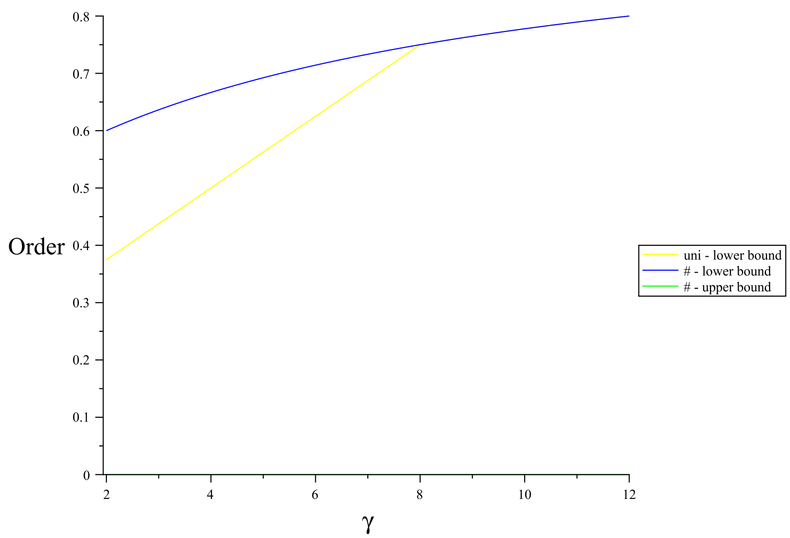

Figure 3.19: Order of convergence in the case (TC), $d = 2$, $\alpha = 6$ and $\beta = 8$

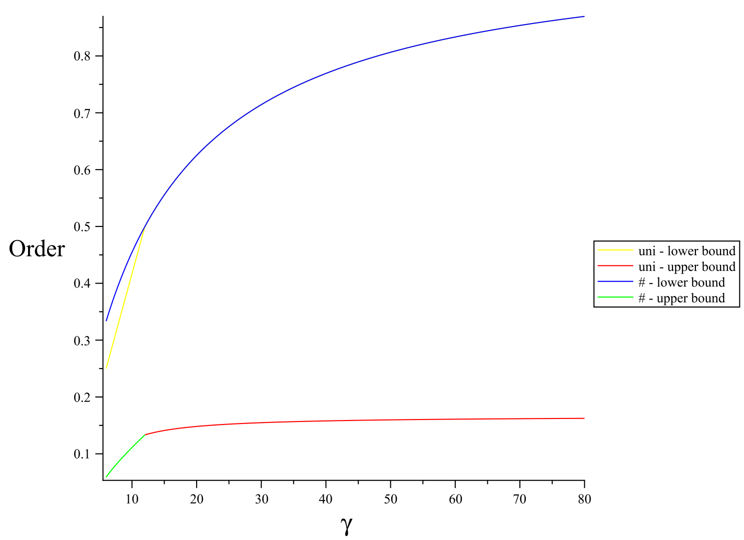

Figure 3.20: Order of convergence in the case (TC), $d = 6$, $\alpha = 6$ and $\beta = 2$

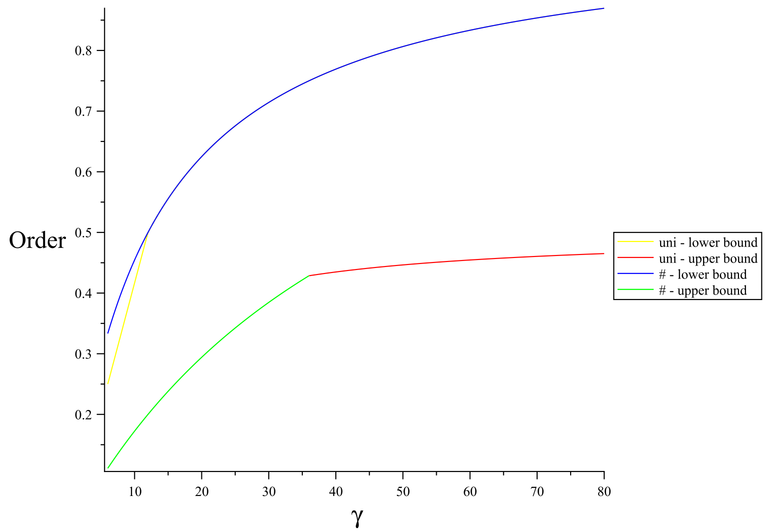

Figure 3.21: Order of convergence in the case (TC), $d = 6$, $\alpha = 6$ and $\beta = 6$

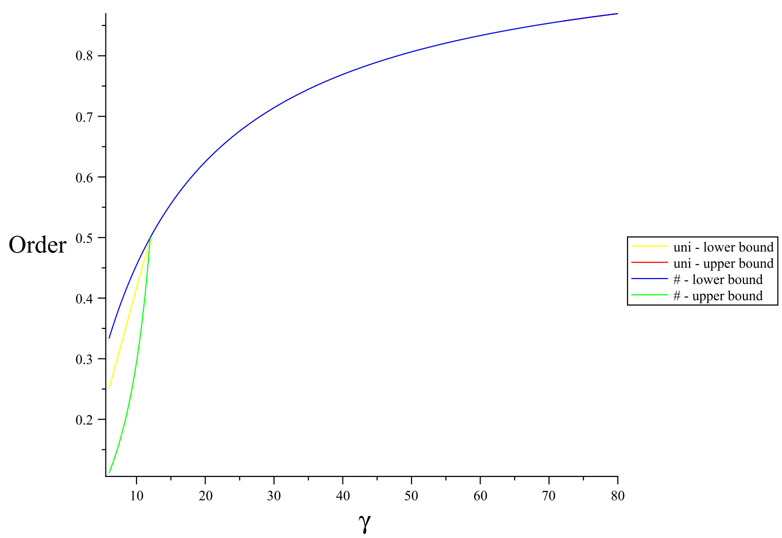

Figure 3.22: Order of convergence in the case (TC), $d = 6$, $\alpha = 6$ and $\beta = 12$

- $\max(\alpha,\gamma) \leq \beta$ and $\gamma - \alpha < d - \epsilon$,
- $\alpha \leq \beta \leq \gamma$ and $\gamma - \alpha < d - \epsilon$,
- $\gamma \leq \beta \leq \alpha$ and $2(\alpha + d)(\gamma + \beta - d) > (\gamma + \alpha - d)(\gamma + \alpha + d) + \epsilon$,
- $\beta \leq \min(\alpha,\gamma)$ and $2(\alpha + d)(\gamma + \beta - d) > (\gamma + \alpha - d)(\gamma + \beta + d) + \epsilon$.

In the case that additionally $\eta + \alpha < 3d$, we get superiority if furthermore

- $\max(\alpha,\gamma) \leq \beta$ and $\gamma(d-1) + \alpha(d+1) > (3d-1)d + \epsilon$ for an arbitrary small $\epsilon > 0$.

Finally, in the additionally limiting case $\eta + \alpha = 3d$, for superiority we need one of the following terms for an arbitrary small $\epsilon > 0$.

- $\max(\alpha,\gamma) \leq \beta$ and $\alpha > d + \epsilon$,
- $\alpha \leq \beta \leq \gamma$ and $\gamma - \alpha < d - \epsilon$,
- $\gamma \leq \beta \leq \alpha$ and $\alpha(2d + \beta - \alpha) > 2d^2 + \epsilon$.

Refer to Section 2.4 for examples of stochastic evolution equations, which fulfil the requirements of Theorem 3.3.1 and 3.3.2, i.e. stochastic heat equations with a multiplication operator as diffusion term.

3.4 Proofs

First, we proof Theorem 3.2.1. For this purpose, we state the following proposition about the cost and the upper bounds of the error of the approximations constructed in Section 3.2.

Proposition 3.4.1

$$\widehat{X}_N^*(T) \in \mathfrak{X}_{c\cdot N}^*, \quad \widehat{X}_N^{\#}(T) \in \mathfrak{X}_{c\cdot N}^{\#}, \quad \widehat{X}_N^{\mathrm{equi}}(T) \in \mathfrak{X}_{c\cdot N}^{\mathrm{equi}} \quad \textit{and} \quad \widehat{X}_N^{\mathrm{uni}}(T) \in \mathfrak{X}_{c\cdot N}^{\mathrm{uni}}$$

for some constant $c > 0$, that only depends on the fixed parameters d, α, γ, p and q.

If furthermore

$$\langle \xi, h_i \rangle^2 \preceq \lambda_i \tag{3.83}$$

for every $i \in \mathbb{N}^d$, then

$$\mathrm{e}\left(\widehat{X}_N^*(T)\right) \preceq \begin{cases} N^{-(\gamma+\alpha-d)/(2d)}, & \textit{if} \quad \gamma+\alpha < 3d, \\ N^{-1} \cdot (\ln N)^{3/2}, & \textit{if} \quad \gamma+\alpha = 3d, \\ N^{-1}, & \textit{if} \quad \gamma+\alpha > 3d, \end{cases} \tag{3.84}$$

$$\mathrm{e}\left(\widehat{X}_N^{\#}(T)\right) \preceq N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)}, \tag{3.85}$$

$$\mathrm{e}\left(\widehat{X}_N^{\mathrm{equi}}(T)\right) \preceq \begin{cases} N^{-(\gamma+\alpha-d)/(2(\alpha+d))}, & \textit{if} \quad \gamma-\alpha < 3d, \\ N^{-1} \cdot (\ln N)^{3/2}, & \textit{if} \quad \gamma-\alpha = 3d, \\ N^{-1}, & \textit{if} \quad \gamma-\alpha > 3d, \end{cases} \tag{3.86}$$

$$\mathrm{e}\left(\widehat{X}_N^{\mathrm{uni}}(T)\right) \preceq \begin{cases} N^{-(\gamma+\alpha-d)/(2(\alpha+d))}, & \textit{if} \quad \gamma-\alpha < d, \\ N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot (\ln N)^{1/2}, & \textit{if} \quad \gamma-\alpha = d, \\ N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)}, & \textit{if} \quad \gamma-\alpha > d. \end{cases} \tag{3.87}$$

Here it is sufficient for the weak asymptotic results to consider algorithms $\widehat{X}_N^{\diamond}(T) \in \mathfrak{X}_{c\cdot N}^{\diamond}$, where $\diamond \in \{*, \#, \mathrm{equi}, \mathrm{uni}\}$, with a constant $c > 0$, which only depends on d, $(\lambda_i)_{i\in\mathbb{N}^d}$, $(\mu_i)_{i\in\mathbb{N}^d}$, p, q, ξ and T.

Proof of Proposition 3.4.1

First, we verify that the constructed algorithms are in the respective stated classes. We have

$$\begin{aligned} \operatorname{cost}\left(\widehat{X}_N^{\mathrm{uni}}(T)\right) &\leq n^{\mathrm{uni}} \cdot \left|\mathcal{I}_N^{\mathrm{uni}}\right| \\ &\preceq \begin{cases} N^{\alpha/(\alpha+d)} \cdot N^{d/(\alpha+d)}, & \text{if } \gamma-\alpha < d, \\ N^{(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot N^{2d/(\gamma+\alpha+d)}, & \text{if } \gamma-\alpha \geq d, \end{cases} \\ &\preceq N \end{aligned}$$

and

$$\operatorname{cost}\left(\widehat{X}_N^{\#}(T)\right) \leq n^{\#} \cdot \left|\mathcal{I}_N^{\#}\right| \preceq N^{(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot N^{(2d)/(\gamma+\alpha+d)} \preceq N.$$

Furthermore, use Lemma C.0.3 to obtain

$$
\begin{aligned}
\mathrm{cost}\left(\widehat{X}_N^{\mathrm{equi}}(T)\right) &\leq \sum_{i \in \mathcal{I}_N^{\mathrm{equi}}} n_i^{\mathrm{equi}} \\
&\preceq \begin{cases} \sum_{|i|_2 \leq N^{1/(\alpha+d)}} (\lambda_i/\mu_i)^q \cdot N^{(\alpha+(\gamma+\alpha)q)/(\alpha+d)}, & \text{if } \gamma-\alpha < 3d, \\ \sum_{|i|_2 \leq N^{2/(\gamma+\alpha-d)}} (\lambda_i/\mu_i)^q \cdot N/\ln(N), & \text{if } \gamma-\alpha = 3d, \\ \sum_{|i|_2 \leq N^{2/(\gamma+\alpha-d)}} (\lambda_i/\mu_i)^q \cdot N, & \text{if } \gamma-\alpha > 3d, \end{cases} \\
&\preceq \begin{cases} N^{(\alpha+(\gamma+\alpha)q)/(\alpha+d)} \cdot \int_1^{N^{1/(\alpha+d)}} x^{-(\gamma+\alpha)q+d-1}\,dx, & \text{if } \gamma-\alpha < 3d, \\ N/\ln(N) \cdot \int_1^{N^{2/(\gamma+\alpha-d)}} x^{-(\gamma+\alpha)q+d-1}\,dx, & \text{if } \gamma-\alpha = 3d, \\ N \cdot \int_1^{N^{2/(\gamma+\alpha-d)}} x^{-(\gamma+\alpha)q+d-1}\,dx, & \text{if } \gamma-\alpha > 3d, \end{cases} \\
&\preceq \begin{cases} N^{(\alpha+(\gamma+\alpha)q)/(\alpha+d)} \cdot N^{(-(\gamma+\alpha)q+d)/(\alpha+d)}, & \text{if } \gamma-\alpha < 3d, \\ N/\ln(N) \cdot \ln(N^{2/(\gamma+\alpha-d)}), & \text{if } \gamma-\alpha = 3d, \\ N, & \text{if } \gamma-\alpha > 3d, \end{cases} \\
&\preceq N
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{cost}\left(\widehat{X}_N^{*}(T)\right) &\leq \sum_{i \in \mathcal{I}_N^{*}} n_i^{*} \\
&\preceq \begin{cases} \sum_{|i|_2 \leq N^{1/d}} (\lambda_i/\mu_i)^p \cdot N^{((\gamma+\alpha)p)/d}, & \text{if } \gamma+\alpha < 3d, \\ \sum_{|i|_2 \leq N^{1/d}} (\lambda_i/\mu_i)^p \cdot N/\ln(N), & \text{if } \gamma+\alpha = 3d, \\ \sum_{|i|_2 \leq N^{2/(\gamma+\alpha-d)}} (\lambda_i/\mu_i)^p \cdot N, & \text{if } \gamma+\alpha > 3d, \end{cases} \\
&\preceq \begin{cases} N^{((\gamma+\alpha)p)/d} \cdot \int_1^{N^{1/d}} x^{-(\gamma+\alpha)p+d-1}\,dx, & \text{if } \gamma+\alpha < 3d, \\ N/\ln(N) \cdot \int_1^{N^{1/d}} x^{-(\gamma+\alpha)p+d-1}\,dx, & \text{if } \gamma+\alpha = 3d, \\ N \cdot \int_1^{N^{2/(\gamma+\alpha-d)}} x^{-(\gamma+\alpha)p+d-1}\,dx, & \text{if } \gamma+\alpha > 3d, \end{cases} \\
&\preceq \begin{cases} N^{((\gamma+\alpha)p)/d} \cdot N^{(-(\gamma+\alpha)p+d)/d}, & \text{if } \gamma+\alpha < 3d, \\ N/\ln(N) \cdot \ln(N^{1/d}), & \text{if } \gamma+\alpha = 3d, \\ N, & \text{if } \gamma+\alpha > 3d, \end{cases} \\
&\preceq N.
\end{aligned}
$$

Hence, all the algorithms are in the stated classes. Now, we determine the errors of these algorithms. For this purpose, note that for any algorithm $\widehat{X}_N(T) \in \mathfrak{X}_N^*$ of the form (3.17) approximating the solution (3.14), the Parseval equality and the continuity of the scalar product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ give

$$
\begin{aligned}
\mathrm{e}^2\left(\widehat{X}_N(T)\right) &= \mathrm{E}\left\|X(T) - \widehat{X}_N(T)\right\|^2 \\
&= \mathrm{E}\Bigg\|\sum_{i\in\mathbb{N}^d}\left(\exp(-\mu_i T)\cdot\langle \xi, h_i\rangle + \lambda_i^{1/2}\cdot Y_i(T)\right)\cdot h_i \\
&\qquad - \sum_{i\in\mathcal{I}_N}\left(\exp(-\mu_i T)\cdot\langle \xi, h_i\rangle + \lambda_i^{1/2}\cdot \widehat{Y}_{i,N}(T)\right)\cdot h_i\Bigg\|^2 \\
&= \mathrm{E}\sum_{k\in\mathbb{N}^d}\Bigg(\sum_{i\notin\mathcal{I}_N}\left(\exp(\mu_i T)\cdot\langle \xi, h_i\rangle + \lambda_i^{1/2}\cdot Y_i(T)\right)\cdot\langle h_i, h_k\rangle \\
&\qquad + \sum_{i\in\mathcal{I}_N}\lambda_i^{1/2}\cdot\left(Y_i(T) - \widehat{Y}_{i,N}(T)\right)\cdot\langle h_i, h_k\rangle\Bigg)^2,
\end{aligned}
$$

where the exchange of the summation and the scalar product is made by considering finite sums and then passing to the limit. Recall the definition (3.18) of the Euler-Maruyama scheme, which implies

$$
\widehat{Y}_{i,N}(T) = \sum_{k=0}^{n_i-1}\Delta_{k,i}\beta_i\prod_{\ell=k}^{n_i-1}\left(1+\mu_i\cdot\Delta_{\ell,i}\right)^{-1}. \tag{3.88}
$$

Using $E(Y_i(T)) = 0$ for every $i \in \mathbb{N}^d$ and $\mathrm{E}(\widehat{Y}_{i,N}(T)) = 0$ for every $i \in \mathcal{I}_N$ as well as that $(\beta_i)_{i\in\mathbb{N}^d}$ is an independent family of scalar Brownian motions and $\langle h_i, h_k\rangle \cdot \langle h_j, h_k\rangle = 0$ for every $k \in \mathbb{N}^d$ if $i \neq j$, we conclude for any of the constructed approximations

$$
\begin{aligned}
\mathrm{e}^2\left(\widehat{X}_N(T)\right) &= \sum_{i\notin\mathcal{I}_N}\exp(-2\mu_i T)\cdot\langle \xi, h_i\rangle^2 \\
&\quad + \sum_{i\in\mathcal{I}_N}\lambda_i\cdot\mathrm{E}\left(Y_i(T) - \widehat{Y}_{i,N}(T)\right)^2 + \sum_{i\notin\mathcal{I}_N}\lambda_i\cdot\mathrm{E}Y_i^2(T).
\end{aligned} \tag{3.89}
$$

We can estimate the summands in the first series by using $\exp(-x) < 1/x$ for $x > 0$ and (3.83) to obtain

$$\exp(-2\mu_i T) \cdot \langle \xi, h_i \rangle^2 \preceq \frac{\lambda_i}{\mu_i} \tag{3.90}$$

for $i \in \mathbb{N}^d$. For the estimation of the summands of the third series, we use (3.15) and the Ito isometry to get

$$\mathrm{E}Y^2(T) = \int_0^T \exp(-2\mu_i(T-t)\,dt \preceq \frac{1}{\mu_i} \tag{3.91}$$

for $i \in \mathbb{N}^d$. To estimate the summands of the second series, we consider the approximation $\widehat{Y}_{i,N}(T)$ with the special choice of regular time nodes satisfying

$$\int_0^{t_{k,i}} \exp(-\mu_i/3 \cdot (T-t))\,dt = \frac{k}{n_i} \cdot \int_0^T \exp(-\mu_i/3 \cdot (T-t))\,dt \tag{3.92}$$

for $k = 0, \ldots, n_i$ and $i \in \mathcal{I}_N$. Then using (3.15), (3.88) and the Ito isometry yields

$$\begin{aligned}
&\mathrm{E}\left(Y_i(T) - \widehat{Y}_{i,N}(T)\right)^2 \\
&= \sum_{k=0}^{n_i-1} \int_{t_{k,i}}^{t_{k+1,i}} \left(\exp(-\mu_i(T-t) - \prod_{\ell=k}^{n_i-1}(1+\mu_i\Delta_{\ell,i})^{-1}\right)^2 dt \\
&\leq 2 \cdot \left(\sum_{k=0}^{n_i-1} (\exp(-\mu_i(T-t)) - \exp(-\mu_i(t-t_{k,i})))^2\,dt \right.\\
&\quad \left. + \sum_{k=0}^{n_i-1} \int_{t_{k,i}}^{t_{k+1,i}} \left(\exp(-\mu_i(T-t_{k,i})) - \prod_{\ell=k}^{n_i-1}(1-\mu_i\Delta_{\ell,i})^{-1}\right)^2 dt \right).
\end{aligned}$$

Thus, Lemma C.0.7 implies

$$\mathrm{E}\left(Y_i(T) - \widehat{Y}_{i,N}(T)\right)^2 \preceq \frac{1}{\mu_i n_i^2} \tag{3.93}$$

for $i \in \mathcal{I}_N$. Now, we consider the approximation $\widehat{Y}^{\mathrm{equi}}_{i,N}(T)$ for $i \in \mathcal{I}_N$. Here we assume without loss of generality that N is sufficiently large, such that $n_i \geq \max(\mu_i, T)$ for every $i \in \mathcal{I}_N$, because in (3.23) up to (3.28) the parameters are chosen in a way that

$n_i^{\mathrm{equi}} \succeq \mu_i$ for every $i \in \mathcal{I}_N^{\mathrm{equi}}$ and $n^{\mathrm{uni}} \succeq \mu_i$ for every $i \in \mathcal{I}_N^{\mathrm{uni}}$. Moreover, inserting the equidistant time nodes $t_{k,i} = k/n_i \cdot T$, $k = 0, \ldots, n_i$ in (3.88) yields

$$\widehat{Y}_{i,N}^{\mathrm{equi}}(T) = \sum_{k=0}^{n_i-1} \left(\beta_i(\frac{k+1}{n_i}T) - \beta_i(\frac{k}{n_i}T) \right) \prod_{\ell=k}^{n_i-1} \left(1 + \mu_i \cdot \frac{1}{n_i} T \right)^{-1}. \tag{3.94}$$

Then it follows from (3.15), (3.94), the Ito isometry and Lemma C.0.6, that

$$\begin{aligned} \mathrm{E}\left(Y_i(T) - \widehat{Y}_{i,N}^{\mathrm{equi}}(T) \right)^2 &= \sum_{k=0}^{n_i-1} \int_{\frac{k}{n_i}T}^{\frac{k+1}{n_i}T} \left(\exp(-\mu_i(T-t)) - \prod_{\ell=k}^{n_i-1} \left(1 + \mu_i \frac{T}{n_i} \right)^{-1} \right)^2 dt \\ &\preceq \frac{\mu_i}{n_i^2} \end{aligned} \tag{3.95}$$

for $i \in \mathcal{I}_N$. We apply (3.90), (3.91) and (3.93) or (3.95) in (3.89) to obtain

$$\mathrm{e}\left(\widehat{X}_N(T) \right) \preceq \sum_{i \in \mathcal{I}_N} \frac{\lambda_i}{\mu_i n_i^2} + \sum_{i \notin \mathcal{I}_N} \frac{\lambda_i}{\mu_i} \tag{3.96}$$

for every algorithm $\widehat{X}_N(T) \in \mathfrak{X}_N^*$ that uses the drift-implicit Euler-Maruyama scheme (3.18) with the considered regularly generated time discretization (3.92) and

$$\mathrm{e}\left(\widehat{X}_N(T) \right) \preceq \sum_{i \in \mathcal{I}_N} \frac{\lambda_i \mu_i}{n_i^2} + \sum_{i \notin \mathcal{I}_N} \frac{\lambda_i}{\mu_i} \tag{3.97}$$

for every algorithm $\widehat{X}_N(T) \in \mathfrak{X}_N^{\mathrm{equi}}$ using (3.18) with equidistant time nodes.

Now, we insert $\widehat{X}_N^*(T)$ with $\mathcal{I}_N^*$ and n_i^* as well as $\widehat{X}_N^\#(T)$ with $\mathcal{I}_N^\#$ and $n^\#$ in (3.96) to obtain by Lemma C.0.3,

$$
\begin{aligned}
\mathrm{e}\left(\widehat{X}_N^*(T)\right)^2 &\preceq \begin{cases} N^{-2(\gamma+\alpha)p/d} \cdot \sum\limits_{|i|_2 \le N^{1/d}} (\lambda_i/\mu_i)^{1-2p} + \sum\limits_{|i|_2 > N^{1/d}} (\lambda_i/\mu_i), & \text{if } \gamma+\alpha < 3d, \\ N^{-2} \cdot (\ln(N))^2 \cdot \sum\limits_{|i|_2 \le N^{1/d}} (\lambda_i/\mu_i)^{1-2p} + \sum\limits_{|i|_2 > N^{1/d}} (\lambda_i/\mu_i), & \text{if } \gamma+\alpha = 3d, \\ N^{-2} \cdot \sum\limits_{|i|_2 \le N^{2/(\gamma+\alpha-d)}} (\lambda_i/\mu_i)^{1-2p} + \sum\limits_{|i|_2 > N^{2/(\gamma+\alpha-d)}} (\lambda_i/\mu_i), & \text{if } \gamma+\alpha > 3d, \end{cases} \\
&\preceq \begin{cases} N^{-2(\gamma+\alpha)p/d} \cdot \int\limits_1^{N^{1/d}} x^{-(\gamma+\alpha)(1-2p)+d-1}\, dx + \int\limits_{N^{1/d}}^{\infty} x^{-(\gamma+\alpha)+d-1}\, dx, & \text{if } \gamma+\alpha < 3d, \\ N^{-2} \cdot (\ln(N))^2 \cdot \int\limits_1^{N^{1/d}} x^{-(\gamma+\alpha)(1-2p)+d-1}\, dx + \int\limits_{N^{1/d}}^{\infty} x^{-(\gamma+\alpha)+d-1}\, dx, & \text{if } \gamma+\alpha = 3d, \\ N^{-2} \cdot \int\limits_1^{N^{2/(\gamma+\alpha-d)}} x^{-(\gamma+\alpha)(1-2p)+d-1}\, dx + \int\limits_{N^{2/(\gamma+\alpha-d)}}^{\infty} x^{-(\gamma+\alpha)+d-1}\, dx, & \text{if } \gamma+\alpha > 3d, \end{cases} \\
&\preceq \begin{cases} N^{-2(\gamma+\alpha)p/d} \cdot N^{(-(\gamma+\alpha)(1-2p)+d)/d} + N^{(-(\gamma+\alpha)+d)/d}, & \text{if } \gamma+\alpha < 3d, \\ N^{-2} \cdot (\ln(N))^3 + N^{(-(\gamma+\alpha)+d)/d}, & \text{if } \gamma+\alpha = 3d, \\ N^{-2} + N^{(-2(\gamma+\alpha-d))/(\gamma+\alpha-d)}, & \text{if } \gamma+\alpha > 3d, \end{cases} \\
&\preceq \begin{cases} N^{-(\gamma+\alpha-d)/d}, & \text{if } \gamma+\alpha < 3d, \\ N^{-2} \cdot (\ln(N))^3, & \text{if } \gamma+\alpha = 3d, \\ N^{-2}, & \text{if } \gamma+\alpha > 3d, \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{e}\left(\widehat{X}_N^\#(T)\right)^2 \preceq N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot \sum_{|i|_2 \le N^{2/(\gamma+\alpha+d)}} (\lambda_i/\mu_i) + \sum_{|i|_2 > N^{2/(\gamma+\alpha+d)}} (\lambda_i/\mu_i) \\
\preceq \quad N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot \int_1^{N^{2/(\gamma+\alpha+d)}} x^{-(\gamma+\alpha)+d-1}\, dx + \int_{N^{2/(\gamma+\alpha+d)}}^{\infty} x^{-(\gamma+\alpha)+d-1}\, dx \\
\preceq \quad N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)}.
\end{aligned}
$$

Finally, we insert $\widehat{X}_N^{\text{equi}}$ with $\mathcal{I}_N^{\text{equi}}$ and n_i^{equi} as well as $\widehat{X}_N^{\text{uni}}$ with $\mathcal{I}_N^{\text{uni}}$ and n^{uni} in (3.97) to derive with Lemma C.0.3 the errors

$$
\begin{aligned}
&\mathrm{e}\left(\widehat{X}_N^{\text{equi}}(T)\right)^2 \\
&\preceq \begin{cases}
N^{-2(\alpha+(\gamma+\alpha)q)/(\alpha+d)} \cdot \sum\limits_{|i|_2 \le N^{1/(\alpha+d)}} \lambda_i^{1-2q}\mu_i^{1+2q} + \sum\limits_{|i|_2 > N^{1/(\alpha+d)}} (\lambda_i/\mu_i), & \text{if } \gamma-\alpha < 3d, \\
N^{-2} \cdot (\ln(N))^2 \cdot \sum\limits_{|i|_2 \le N^{2/(\gamma+\alpha-d)}} \lambda_i^{1-2q}\mu_i^{1+2q} + \sum\limits_{|i|_2 > N^{2/(\gamma+\alpha-d)}} (\lambda_i/\mu_i), & \text{if } \gamma-\alpha = 3d, \\
N^{-2} \cdot \sum\limits_{|i|_2 \le N^{2/(\gamma+\alpha-d)}} \lambda_i^{1-2q}\mu_i^{1+2q} + \sum\limits_{|i|_2 > N^{2/(\gamma+\alpha-d)}} (\lambda_i/\mu_i), & \text{if } \gamma-\alpha > 3d,
\end{cases} \\
&\preceq \begin{cases}
N^{-2(\alpha+(\gamma+\alpha)q)/(\alpha+d)} \cdot \int\limits_1^{N^{1/(\alpha+d)}} x^{\gamma(2q-1)+\alpha(2q+1)+d-1}\,dx + \\
\qquad + \int\limits_{N^{1/(\alpha+d)}}^{\infty} x^{-(\gamma+\alpha)+d-1}, & \text{if } \gamma-\alpha < 3d, \\
N^{-2} \cdot (\ln(N))^2 \cdot \int\limits_1^{N^{2/(\gamma+\alpha-d)}} x^{\gamma(2q-1)+\alpha(2q+1)+d-1}\,dx + \\
\qquad + \int\limits_{N^{2/(\gamma+\alpha-d)}}^{\infty} x^{-(\gamma+\alpha)+d-1}\,dx, & \text{if } \gamma-\alpha = 3d, \\
N^{-2} \cdot \int\limits_1^{N^{2/(\gamma+\alpha-d)}} x^{\gamma(2q-1)+\alpha(2q+1)+d-1}\,dx + \\
\qquad + \int\limits_{N^{2/(\gamma+\alpha-d)}}^{\infty} x^{-(\gamma+\alpha)+d-1}, & \text{if } \gamma-\alpha > 3d,
\end{cases} \\
&\preceq \begin{cases}
N^{-2(\alpha+(\gamma+\alpha)q)/(\alpha+d)} \cdot N^{(\gamma(2q-1)+\alpha(2q+1)+d)/(\alpha+d)} + \\
\qquad + N^{(-(\gamma+\alpha)+d)/(\alpha+d)}, & \text{if } \gamma-\alpha < 3d, \\
N^{-2} \cdot (\ln(N))^3 + N^{-2(\gamma+\alpha-d)/(\gamma+\alpha-d)}, & \text{if } \gamma-\alpha = 3d, \\
N^{-2} + N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)}, & \text{if } \gamma-\alpha > 3d,
\end{cases} \\
&\preceq \begin{cases}
N^{-(\gamma+\alpha-d)/(\alpha+d)}, & \text{if } \gamma-\alpha < 3d, \\
N^{-2} \cdot (\ln(N))^3, & \text{if } \gamma-\alpha = 3d, \\
N^{-2}, & \text{if } \gamma-\alpha > 3d,
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{e}\left(\widehat{X}_N^{\mathrm{uni}}(T)\right)^2 \\
&\preceq \begin{cases} N^{-2\alpha/(\alpha+d)} \cdot \sum\limits_{|i|_2 \le N^{1/(\alpha+d)}} \lambda_i\mu_i + \sum\limits_{|i|_2 > N^{1/(\alpha+d)}} (\lambda_i/\mu_i), & \text{if } \gamma-\alpha < d, \\ N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot \sum\limits_{|i|_2 \le N^{2/(\gamma+\alpha+d)}} \lambda_i\mu_i + \sum\limits_{|i|_2 > N^{2/(\gamma+\alpha+d)}} (\lambda_i/\mu_i), & \text{if } \gamma-\alpha \ge d, \end{cases} \\
&\preceq \begin{cases} N^{-2\alpha/(\alpha+d)} \cdot \int\limits_1^{N^{1/(\alpha+d)}} x^{-\gamma+\alpha+d-1}\, dx + \\ \qquad + \int\limits_{N^{1/(\alpha+d)}}^{\infty} x^{-(\gamma+\alpha)+d-1}\, dx, \quad \text{if } \gamma-\alpha < d, \\ N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot \int\limits_1^{N^{2/(\gamma+\alpha+d)}} x^{-\gamma+\alpha+d-1}\, dx + \\ \qquad + \int\limits_{N^{2/(\gamma+\alpha+d)}}^{\infty} x^{-(\gamma+\alpha)+d-1}, \quad \text{if } \gamma-\alpha \ge d, \end{cases} \\
&\preceq \begin{cases} N^{-2\alpha/(\alpha+d)} \cdot N^{(-\gamma+\alpha+d)/(\alpha+d)} + N^{(-(\gamma+\alpha)+d)/(\alpha+d)}, & \text{if } \gamma-\alpha < d, \\ N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot \ln(N) + N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)}, & \text{if } \gamma-\alpha = d, \\ N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)} + N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)}, & \text{if } \gamma-\alpha > d, \end{cases} \\
&\preceq \begin{cases} N^{-(\gamma+\alpha-d)/(\alpha+d)}, & \text{if } \gamma-\alpha < d, \\ N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot \ln(N), & \text{if } \gamma-\alpha = d, \\ N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)}, & \text{if } \gamma-\alpha > d, \end{cases}
\end{aligned}
$$

which finishes the proof. □

To proof Theorem 3.2.1, we also derive lower bounds for the minimal errors of every algorithm $\widehat{X}(T) \in \mathfrak{X}_N^*$, $\widehat{X}(T) \in \mathfrak{X}_N^\#$, $\widehat{X}(T) \in \mathfrak{X}_N^{\mathrm{equi}}$ and $\widehat{X}(T) \in \mathfrak{X}_N^{\mathrm{uni}}$. We obtain the following result.

Proposition 3.4.2

$$\mathrm{e}_N^* \succeq \begin{cases} N^{-(\gamma+\alpha-d)/(2d)}, & \textit{if} \quad \gamma+\alpha < 3d, \\ N^{-1}\cdot(\ln N)^{3/2}, & \textit{if} \quad \gamma+\alpha = 3d, \\ N^{-1}, & \textit{if} \quad \gamma+\alpha > 3d, \end{cases} \tag{3.98}$$

$$\mathrm{e}_N^{\#} \succeq N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)}, \tag{3.99}$$

$$\mathrm{e}_N^{\mathrm{equi}} \succeq \begin{cases} N^{-(\gamma+\alpha-d)/(2(\alpha+d))}, & \textit{if} \quad \gamma-\alpha < 3d, \\ N^{-1}\cdot(\ln N)^{3/2}, & \textit{if} \quad \gamma-\alpha = 3d, \\ N^{-1}, & \textit{if} \quad \gamma-\alpha > 3d, \end{cases} \tag{3.100}$$

$$\mathrm{e}_N^{\mathrm{uni}} \succeq \begin{cases} N^{-(\gamma+\alpha-d)/(2(\alpha+d))}, & \textit{if} \quad \gamma-\alpha < d, \\ N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)}\cdot(\ln N)^{1/2}, & \textit{if} \quad \gamma-\alpha = d, \\ N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)}, & \textit{if} \quad \gamma-\alpha > d. \end{cases} \tag{3.101}$$

Proof of Proposition 3.4.2

Step 1: Lower error bounds for any algorithm of the classes.

First, we consider any approximation $\widehat{X}(T) \in \mathfrak{X}_N^*$ of the solution $X(T)$ given by (3.14). For the error of such an algorithm, we have

$$\begin{aligned} &\mathrm{E}\left\|X(T)-\widehat{X}(T)\right\|^2 \\ &= \mathrm{E}\left\|\sum_{i\in\mathbb{N}^d}\left(\exp(-\mu_i T)\cdot\langle\xi,h_i\rangle+\lambda_i^{1/2}\cdot Y_i(T)\right)\cdot h_i-\widehat{X}(T)\right\|^2. \end{aligned} \tag{3.102}$$

Given a vector $i \in \mathbb{N}^d$ of integers, fixed time nodes $(t_{k,i})_{k\le n_i}$ in $[0,T]$ with $n_i \in \mathbb{N}$ and the evaluations $\beta_i(t_{1,i}),\ldots,\beta_i(t_{n_i,i})$, then we know that the conditional expectation

$$\widehat{Y}_i(T) = \mathrm{E}(Y_i(T)\,|\,\beta_i(t_1,i),\ldots,\beta_i(t_{n_i,i})) \tag{3.103}$$

is the best choice for an approximation of $Y_i(T)$. Therefore, with an arbitrarily chosen non-empty, finite set $\mathcal{I} \subset \mathbb{N}^d$, a sequence $(n_i)_{i\in\mathcal{I}}$ and a time discretization $(t_{k,i})_{k\le n_i, i\in\mathcal{I}}$

of $[0,T]$, the best choice of $\widehat{X}(T)$ is of the form

$$\begin{aligned}\widehat{X}^*(T) &= \sum_{i\in\mathcal{I}}\Big(\exp(-\mu_i T)\cdot\langle\xi,h_i\rangle + \lambda_i^{1/2}\cdot\widehat{Y}_i(N)\Big)\cdot h_i \\ &\quad + \sum_{i\notin\mathcal{I}}\exp(-\mu_i T)\cdot\langle\xi,h_i\rangle\cdot h_i. \qquad (3.104)\end{aligned}$$

Note, that the conditional expectation $\widehat{\beta}_i$ of a scalar Brownian motion $\beta_i = (\beta_i(t))_{t\geq 0}$, given its evaluations at the time nodes $(t_{k,i})_{k\leq n_i}$, is derived by piecewise linear interpolation, i.e. for $t\in[t_{k,i},t_{k+1,i}]$,

$$\begin{aligned}\widehat{\beta}_i(t) &= \mathrm{E}(\beta_i(t)\,|\,\beta_i(t_{1,i}),\ldots,\beta_i(t_{n_i,i})) \\ &= \beta_i(t_{k,i}) + \frac{t-t_{k,i}}{t_{k+1,i}-t_{k,i}}\cdot(\beta_i(t_{k+1,i})-\beta_i(t_{k,i})).\end{aligned}$$

In addition, $(\beta_i)_{i\in\mathbb{N}^d}$ is an independent family of scalar Brownian motions and using Lemma C.0.1, we have for $i\in\mathbb{N}^d$,

$$\begin{aligned}Y_i(T)-\widehat{Y}_i(T) &= \beta_i(T) - \mu_i\int_0^T \exp(-\mu_i(T-t))\cdot\beta_i(t)\,dt \\ &\quad -\widehat{\beta}_i(T) + \mu_i\int_0^T\exp(-\mu_i(T-t))\cdot\widehat{\beta}_i(t)\,dt. \qquad (3.105)\end{aligned}$$

Hence, we obtain with $\widehat{X}(T)=\widehat{X}^*(T)$ in (3.102) by the Parseval equality and the continuity of the scalar product

$$\begin{aligned}&\mathrm{E}\left\|X(T)-\widehat{X}(T)\right\|^2 \\ &\geq \mathrm{E}\left\|\sum_{i\in\mathcal{I}}\lambda_i^{1/2}\cdot(Y_i(T)-\widehat{Y}_i(T))\cdot h_i + \sum_{i\notin\mathcal{I}}\lambda_i^{1/2}\cdot Y_i(T)\cdot h_i\right\|^2 \\ &= \mathrm{E}\sum_{k\in\mathbb{N}^d}\left(\sum_{i\in\mathcal{I}}\lambda_i^{1/2}\cdot(Y_i(T)-\widehat{Y}_i(T))\cdot\langle h_i,h_k\rangle + \sum_{i\notin\mathcal{I}}\lambda_i^{1/2}\cdot Y_i(T)\cdot\langle h_i,h_k\rangle\right)^2.\end{aligned}$$

Because of $\mathrm{E}(Y_i(T))=0$ for every $i\in\mathbb{N}^d$ and $\langle h_i,h_k\rangle\cdot\langle h_j,h_k\rangle = 0$ for every $k\in\mathbb{N}^d$ if $i\neq j$, we conclude

$$\mathrm{E}\left\|X(T)-\widehat{X}(T)\right\|^2 \geq \sum_{i\in\mathcal{I}}\lambda_i\cdot\mathrm{E}\left(Y_i(T)-\widehat{Y}_i(T)\right)^2 + \sum_{i\notin\mathcal{I}}\lambda_i\cdot\mathrm{E}Y_i^2(T). \qquad (3.106)$$

From the Ito isometry, we get for $i \in \mathbb{N}^d$,

$$\mathrm{E}Y_i^2(T) = \int_0^T \exp(-2\mu_i(T-t))\,dt \succeq \frac{1}{\mu_i}. \tag{3.107}$$

To estimate $\mathrm{E}\left(Y_i(T) - \widehat{Y}_i(T)\right)^2$, we use Lemma 1 in [MGRW07], which gives

$$\mathrm{E}\left(Y_i(T) - \widehat{Y}_i(T)\right)^2 \succeq \frac{1}{\mu_i n_i^2} \tag{3.108}$$

for $i \in \mathcal{I}$ and for a fixed arbitrary time discretization of $[0,T]$. Inserting (3.107) and (3.108) in (3.106), the lower error bound for $\widehat{X}(T)$ can be estimated by

$$\mathrm{E}\left\|X(T) - \widehat{X}(T)\right\|^2 \succeq \sum_{i\in\mathcal{I}} \frac{\lambda_i}{\mu_i n_i^2} + \sum_{i\notin\mathcal{I}} \frac{\lambda_i}{\mu_i}. \tag{3.109}$$

Now, we turn to the more restrictive class of algorithms using an equidistant time discretization. Thus, let $\widehat{X}^{\mathrm{equi}}(T) \in \mathfrak{X}_N^{\mathrm{equi}}$ be such an algorithm of the form

$$\begin{aligned}\widehat{X}^{\mathrm{equi}}(T) \;&=\; \sum_{i\in\mathcal{I}} \left(\exp(-\mu_i T)\cdot\langle \xi, h_i\rangle + \lambda_i \cdot \widehat{Y}_i^{\mathrm{equi}}(T)\right)\cdot h_i \\ &\quad + \sum_{i\notin\mathcal{I}} \exp(-\mu_i T)\cdot\langle \xi, h_i\rangle \cdot h_i\end{aligned}$$

with

$$\widehat{Y}_i^{\mathrm{equi}}(T) = \mathrm{E}(Y_i(T) \mid \beta_i(1/n_i \cdot T), \ldots, \beta_i(T)),$$

similar to (3.103) and (3.104) with equidistant time nodes $t_{k,i} = k/n_i{\cdot}T$ for $k = 1,\ldots,n_i$ and $i \in \mathcal{I}$. The analogous approach to obtain (3.106) yields

$$\mathrm{E}\left\|X(T) - \widehat{X}^{\mathrm{equi}}(T)\right\|^2 \geq \sum_{i\in\mathcal{I}} \lambda_i \cdot \mathrm{E}\left(Y_i(T) - \widehat{Y}_i^{\mathrm{equi}}(T)\right)^2 + \sum_{i\notin\mathcal{I}} \lambda_i \cdot \mathrm{E}Y_i^2(T). \tag{3.110}$$

To estimate $\mathrm{E}\left(Y_i(T) - \widehat{Y}_i^{\mathrm{equi}}(T)\right)^2$, we use again Lemma 1 in [MGRW07]. Consequently,

$$\mathrm{E}\left(Y_i(T) - \widehat{Y}_i^{\mathrm{equi}}(T)\right)^2 \succeq \min\left(\frac{\mu_i}{n_i^2}, \frac{1}{\mu_i}\right) \tag{3.111}$$

for $i \in \mathcal{I}$. Combining (3.107), (3.110) and (3.111) yields

$$\mathrm{E}\left\|X(T)-\widehat{X}^{\mathrm{equi}}(T)\right\|^2 \succeq \sum_{i\in\mathcal{I}} \min\left(\frac{\lambda_i\mu_i}{n_i^2}, \frac{\lambda_i}{\mu_i}\right) + \sum_{i\notin\mathcal{I}} \frac{\lambda_i}{\mu_i}. \tag{3.112}$$

In [MGRW07] and [MGRW08], with respect to a stochastic heat equation, the authors already analyze the optimization problems for terms of the form as on the right-hand sides in (3.109) and (3.112) taken over $\mathcal{I} \subset \mathbb{N}^d$ and $(n_i)_{i\in\mathcal{I}} \in \mathbb{N}^{\mathcal{I}}$ satisfying the constraint $\sum_{i\in\mathcal{I}} n_i \leq N$.

Step 2: Optimal choice of an index set.

Claim 1: For any $K \in \mathbb{N}$, an index set of the form $\mathcal{I} = \left\{i \in \mathbb{N}^d \,|\, |i|_2 \leq K\right\}$ is optimal.

We show, that this claim holds true for the right-hand side in equation (3.109). For this purpose, let $\mathcal{I} = \left\{i \in \mathbb{N}^d \,|\, |i|_2 \leq K\right\}$ and $\mathcal{J} \subset \mathbb{N}^d$ be a non-empty, finite set with $|\mathcal{J}| \leq |\mathcal{I}|$. Furthermore, for a fixed integer $k \leq |\mathcal{I}|$, we put $\mathcal{V}_k = \{v_1, \ldots, v_k\} \subset \mathcal{I}$, $\mathcal{W}_k = \{w_1, \ldots, w_k\} \subset \mathbb{N}^d \setminus \mathcal{I}$ and $n_{x_\ell} = n_{y_\ell}$ for every $\ell \in \{1, \ldots, k\}$. Now, we prove

$$\sum_{i\in\mathcal{I}} \frac{\lambda_i}{\mu_i n_i^2} + \sum_{i\notin\mathcal{I}} \frac{\lambda_i}{\mu_i} \leq \sum_{i\in\mathcal{J}} \frac{\lambda_i}{\mu_i n_i^2} + \sum_{i\notin\mathcal{J}} \frac{\lambda_i}{\mu_i}. \tag{3.113}$$

If $\mathcal{J} \subset \mathcal{I}$, (3.113) holds true because of $n_i \geq 1$ for every $i \in \mathbb{N}^d$, which yields

$$\sum_{i\in\mathcal{I}\setminus\mathcal{J}} \frac{\lambda_i}{\mu_i n_i} \leq \sum_{i\in\mathcal{I}\setminus\mathcal{J}} \frac{\lambda_i}{\mu_i}.$$

If $\mathcal{J} = (\mathcal{I} \setminus \mathcal{V}_k) \cup \mathcal{W}_k$, we have $\lambda_v/\mu_v \geq \lambda_w/\mu_w$ with $v \in \mathcal{V}_k$ and $w \in \mathcal{W}_k$, which yields

$$\sum_{i\in\mathcal{W}_k} \frac{\lambda_i}{\mu_i}\left(1 - \frac{1}{n_i^2}\right) \leq \sum_{i\in\mathcal{V}_k} \frac{\lambda_i}{\mu_i}\left(1 - \frac{1}{n_i^2}\right).$$

So, Claim 1 holds true and for the right-hand side in equation (3.112), we state a second claim.

Claim 2: $\mathcal{I} = \left\{i \in \mathbb{N}^d \,|\, n_i \geq \mu_i\right\}$ is an optimal choice in (3.112).

We show Claim 2 by contradiction. For this purpose, we put $\mathcal{I} = \{i \in \mathbb{N}^d \,|\, n_i \geq \mu_i\}$ and $\sum_{i\in\emptyset} c_i = 0$ for any real number sequence $(c_i)_{i\in\mathbb{N}^d}$, due to formal reasons. That implies

$$\sum_{i\in\mathcal{I}} \min\left(\frac{\lambda_i\mu_i}{n_i^2}, \frac{\lambda_i}{\mu_i}\right) + \sum_{i\notin\mathcal{I}} \frac{\lambda_i}{\mu_i} = \sum_{i\in\mathcal{I}} \frac{\lambda_i\mu_i}{n_i^2} + \sum_{i\notin\mathcal{I}} \frac{\lambda_i}{\mu_i}.$$

Now, we assume, there exists a set $\mathcal{J} \subset \mathbb{N}^d$ with $\mathcal{J} \neq \mathcal{I}$ satisfying

$$\sum_{i\in\mathcal{I}} \frac{\lambda_i \mu_i}{n_i^2} + \sum_{i\notin\mathcal{I}} \frac{\lambda_i}{\mu_i} > \sum_{i\in\mathcal{J}} \min\left(\frac{\lambda_i\mu_i}{n_i^2}, \frac{\lambda_i}{\mu_i}\right) + \sum_{i\notin\mathcal{J}} \frac{\lambda_i}{\mu_i}.$$

If we rearrange the sum on the right hand side, this means

$$\sum_{i\in\mathcal{I}} \frac{\lambda_i \mu_i}{n_i^2} + \sum_{i\notin\mathcal{I}} \frac{\lambda_i}{\mu_i} > \sum_{i\in\mathcal{J}\cap\mathcal{I}} \frac{\lambda_i\mu_i}{n_i^2} + \sum_{i\notin\mathcal{J}\cap\mathcal{I}} \frac{\lambda_i}{\mu_i}$$

and therefore

$$\sum_{i\in\mathcal{I}\setminus\mathcal{J}} \frac{\lambda_i \mu_i}{n_i^2} > \sum_{i\in\mathcal{I}\setminus\mathcal{J}} \frac{\lambda_i}{\mu_i}.$$

But this is a contradiction to $n_i \geq \mu_i$ for every $i \in \mathcal{I}$, if $\mathcal{I} \setminus \mathcal{J} \neq \emptyset$. In the case, that $\mathcal{I} \setminus \mathcal{J} = \emptyset$, it contradicts $0 = 0$. Thus, Claim 2 is true and an index set of the form

$$\mathcal{I} = \left\{ i \in \mathbb{N}^d \,|\, |i|_2 \preceq n_i^{1/\alpha} \right\}$$

is optimal in (3.112).

Thus, combining Claim 1 and (3.109) gives

$$\mathrm{E} \left\| X(T) - \widehat{X}(T) \right\|^2 \succeq \sum_{|i|_2 \leq K} \frac{\lambda_i}{\mu_i n_i^2} + \sum_{|i|_2 > K} \frac{\lambda_i}{\mu_i} \tag{3.114}$$

for an arbitrary $K \in \mathbb{N}$ and combining Claim 2 and (3.112) gives

$$\mathrm{E} \left\| X(T) - \widehat{X}^{\mathrm{equi}}(T) \right\|^2 \succeq \sum_{|i|_2 \leq K} \frac{\lambda_i \mu_i}{n_i^2} + \sum_{|i|_2 > K} \frac{\lambda_i}{\mu_i} \tag{3.115}$$

for some $K \in \mathbb{N}$ with $K \preceq n_i^{1/\alpha}$ for every $i \in \mathcal{I}$.

Step 3: Calculation of the minimal errors.

Using the Hölder inequality, $\sum_{|i|_2 \le K} n_i \le N$ and Lemma C.0.3 in (3.114) gives

$$
\begin{aligned}
\mathrm{E}\left\|X(T)-\widehat{X}(T)\right\|^2 &\succeq \sum_{|i|_2\le K}\left(\frac{\lambda_i^{1/3}}{\mu_i^{1/3}n_i^{2/3}}\right)^3+\sum_{|i|_2>K}\frac{\lambda_i}{\mu_i} \\
&\ge \left(\sum_{|i|_2\le K}\frac{\lambda_i^{1/3}}{\mu_i^{1/3}n_i^{2/3}}\cdot n_i^{2/3}\right)^3\cdot\left(\sum_{|i|_2\le K}\left(n_i^{2/3}\right)^{3/2}\right)^{-2}+\sum_{|i|_2>K}\frac{\lambda_i}{\mu_i} \\
&\succeq N^{-2}\cdot\left(\int_1^K x^{-(\gamma+\alpha)/3+d-1}\,dx\right)^3+\int_K^\infty x^{-(\gamma+\alpha)+d-1}\,dx \\
&\asymp \begin{cases} N^{-2}\cdot K^{-(\gamma+\alpha)+3d}+K^{-(\gamma+\alpha)+d}, & \text{if } \gamma+\alpha<3d, \\ N^{-2}\cdot(\ln K)^3+K^{-(\gamma+\alpha)+d}, & \text{if } \gamma+\alpha=3d, \\ N^{-2}+K^{-(\gamma+\alpha)+d}, & \text{if } \gamma+\alpha>3d. \end{cases}
\end{aligned}
$$

In the case of $\gamma+\alpha<3d$, we obtain

$$
\begin{aligned}
\mathrm{E}\left\|X(T)-\widehat{X}(T)\right\|^2 &\succeq \begin{cases} N^{-2}\cdot N^{(-(\gamma+\alpha)+3d)/d}, & \text{if } K\ge N^{1/d}, \\ N^{-(\gamma+\alpha-d)/d}, & \text{if } K<N^{1/d}, \end{cases} \\
&= N^{-(\gamma+\alpha-d)/d}.
\end{aligned}
$$

In the case of $\gamma+\alpha=3d$, we obtain

$$
\mathrm{E}\left\|X(T)-\widehat{X}(T)\right\|^2 \succeq N^{-2}\cdot(\ln N)^3, \tag{3.116}
$$

if $K\ge N$. If $K<N$, we consider a constant $c>0$, such that

$$
K = c\cdot N/\ln(K)\le N
$$

to obtain with two further positive constants c_1 and c_2,

$$
\begin{aligned}
\mathrm{E}\left\|X(T)-\widehat{X}(T)\right\|^2 &\succeq N^{-2}\cdot(\ln K)^3+K^{-2d} \\
&\ge N^{-2}\cdot(\ln(c\cdot N)-\ln(\ln(K)))^3 \\
&\succeq N^{-2}\cdot\left(c_1\cdot(\ln(N))^3-c_2\cdot(\ln(\ln(N)))^3\right) \\
&\succeq N^{-2}\cdot(\ln N)^3. \qquad (3.117)
\end{aligned}
$$

Hence, we get

$$(\mathrm{e}_N^*)^2 \succeq \begin{cases} N^{-(\gamma+\alpha-d)/d}, & \text{if } \gamma+\alpha < 3d, \\ N^{-2}\cdot(\ln N)^3, & \text{if } \gamma+\alpha = 3d, \\ N^{-2}, & \text{if } \gamma+\alpha > 3d. \end{cases}$$

Furthermore, if we only consider algorithms $\widehat{X}(T) \in \mathfrak{X}_N^{\#}$, denoted by $\widehat{X}^{\#}(T)$, (3.114) reduces to

$$\mathrm{E}\left\|X(T) - \widehat{X}^{\#}(T)\right\|^2 \succeq n^{-2} \sum_{|i|_2 \le K} \frac{\lambda_i}{\mu_i} + \sum_{|i|_2 > K} \frac{\lambda_i}{\mu_i},$$

because of $n = n_i$ for every $i \in \mathcal{I}$. Now we use $n \cdot |\mathcal{I}| \le N$, $|\mathcal{I}| \asymp K^d$ and Lemma C.0.3, to obtain

$$\begin{aligned} \mathrm{E}\left\|X(T) - \widehat{X}^{\#}(T)\right\|^2 &\succeq N^{-2}\cdot K^{2d} + \int_K^\infty x^{-(\gamma+\alpha)+d-1}\,dx \\ &\succeq N^{-2}\cdot K^{2d} + K^{-(\gamma+\alpha)+d} \\ &\succeq \begin{cases} N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)}, & \text{if } K \ge N^{2/(\gamma+\alpha+d)}, \\ N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)}, & \text{if } K < N^{2/(\gamma+\alpha+d)}. \end{cases} \end{aligned}$$

This yields

$$\left(\mathrm{e}_N^{\#}\right)^2 \succeq N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)}.$$

For the error estimation of the algorithms $\widehat{X}^{\mathrm{equi}}(T) \in \mathfrak{X}_N^{\mathrm{equi}}$, using an equidistant time discretization, we apply the Hölder inequality and Lemma C.0.3 in (3.115), to obtain

$$\begin{aligned} \mathrm{E}\left\|X(T) - \widehat{X}^{\mathrm{equi}}(T)\right\|^2 &\succeq \sum_{|i|_2 \le K} \left(\frac{\lambda_i^{1/3}\mu_i^{1/3}}{n_i^{2/3}}\right)^3 + \sum_{|i|_2 > K} \frac{\lambda_i}{\mu_i} \\ &\ge \left(\sum_{|i|_2 \le K} \frac{\lambda_i^{1/3}\mu_i^{1/3}}{n_i^{2/3}} \cdot n_i^{2/3}\right)^3 \cdot \left(\sum_{|i|_2 \le K} \left(n_i^{2/3}\right)^{3/2}\right)^{-2} + \sum_{|i|_2 > K} \frac{\lambda_i}{\mu_i} \\ &\succeq N^{-2}\cdot\left(\int_1^K x^{-(\gamma-\alpha)/3+d-1}\,dx\right)^3 + \int_K^\infty x^{-(\gamma+\alpha)+d-1}\,dx \\ &\asymp \begin{cases} N^{-2}\cdot K^{-(\gamma-\alpha)+3d} + K^{-(\gamma+\alpha)+d}, & \text{if } \gamma-\alpha < 3d, \\ N^{-2}\cdot(\ln K)^3 + K^{-(\gamma+\alpha)+d}, & \text{if } \gamma-\alpha = 3d, \\ N^{-2} + K^{-(\gamma+\alpha)+d}, & \text{if } \gamma-\alpha > 3d. \end{cases} \end{aligned}$$

In the case of $\gamma - \alpha < 3d$, we have

$$\begin{aligned} \mathrm{E}\left\|X(T) - \widehat{X}^{\mathrm{equi}}(T)\right\|^2 &\succeq \begin{cases} N^{-2} \cdot N^{(-(\gamma-\alpha)+3d)/(\alpha+d)}, & \text{if } K \geq N^{1/(\alpha+d)}, \\ N^{-(\gamma+\alpha-d)/(\alpha+d)}, & \text{if } K < N^{1/(\alpha+d)}, \end{cases} \\ &= N^{-(\gamma+\alpha-d)/(\alpha+d)}. \end{aligned}$$

In the case of $\gamma - \alpha = 3d$, we derive

$$\mathrm{E}\left\|X(T) - \widehat{X}^{\mathrm{equi}}(T)\right\|^2 \succeq N^{-2} \cdot (\ln N)^3$$

in the same way as in (3.116) and (3.117). Thus

$$\left(\mathrm{e}_N^{\mathrm{equi}}\right)^2 \succeq \begin{cases} N^{-(\gamma+\alpha-d)/(\alpha+d)}, & \text{if } \gamma - \alpha < 3d, \\ N^{-2} \cdot (\ln N)^3, & \text{if } \gamma - \alpha = 3d, \\ N^{-2}, & \text{if } \gamma - \alpha > 3d. \end{cases}$$

Finally, let $\widehat{X}^{\mathrm{uni}}(T)$ be an algorithm in the class $\mathfrak{X}_N^{\mathrm{uni}}$. Then, by (3.115) with $n = n_i$ for every $i \in \mathcal{I}$, $n \cdot |\mathcal{I}| \leq N$, $|\mathcal{I}| \asymp K^d$ and Lemma C.0.3, we conclude that

$$\begin{aligned} \mathrm{E}\left\|X(T) - \widehat{X}^{\mathrm{uni}}(T)\right\|^2 &\succeq n^{-2} \sum_{|i|_2 \leq K} \lambda_i \mu_i + \sum_{|i|_2 > K} \frac{\lambda_i}{\mu_i} \\ &\succeq N^{-2} \cdot K^{2d} \cdot \int_1^K x^{-(\gamma-\alpha)+d-1}\,dx + \int_K^\infty x^{-(\gamma+\alpha)+d-1}\,dx \\ &\succeq \begin{cases} N^{-2} \cdot K^{-(\gamma-\alpha)+3d} + K^{-(\gamma+\alpha)+d}, & \text{if } \gamma - \alpha < d, \\ N^{-2} \cdot K^{2d} \cdot (\ln K) + K^{-(\gamma+\alpha)+d}, & \text{if } \gamma - \alpha = d, \\ N^{-2} \cdot K^{2d} + K^{-(\gamma+\alpha)+d}, & \text{if } \gamma - \alpha > d. \end{cases} \end{aligned}$$

If $\gamma - \alpha < d$, we calculate

$$\begin{aligned} \mathrm{E}\left\|X(T) - \widehat{X}^{\mathrm{uni}}(T)\right\|^2 &\succeq \begin{cases} N^{-2} \cdot N^{(-(\gamma-\alpha)+3d)/(\alpha+d)}, & \text{if } K \geq N^{1/(\alpha+d)}, \\ N^{-(\gamma+\alpha-d)/(\alpha+d)}, & \text{if } K < N^{1/(\alpha+d)}, \end{cases} \\ &= N^{-(\gamma+\alpha-d)/(\alpha+d)} \end{aligned}$$

and if $\gamma - \alpha > d$, we derive

$$\begin{aligned} \mathrm{E}\left\| X(T) - \widehat{X}^{\mathrm{uni}}(T) \right\|^2 &\succeq \begin{cases} N^{-2} \cdot N^{4d/(\gamma+\alpha+d)}, & \text{if } K \geq N^{2/(\gamma+\alpha+d)}, \\ N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)}, & \text{if } K < N^{2/(\gamma+\alpha+d)}, \end{cases} \\ &= N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)}. \end{aligned}$$

In the case of $\gamma - \alpha = d$, we obtain immediately

$$\mathrm{E}\left\| X(T) - \widehat{X}^{\mathrm{uni}}(T) \right\|^2 \succeq N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot \ln N$$

for $K \geq N^{2/(\gamma+\alpha+d)}$ and if $K < N^{2/(\gamma+\alpha+d)}$, we note that for a constant $c > 0$ with

$$K = c \cdot N^{2/(\gamma+\alpha+d)}/(\ln N) \leq N^{2/(\gamma+\alpha+d)}$$

this estimation holds. Hence, we have

$$\left(\mathrm{e}_N^{\mathrm{uni}}\right)^2 \succeq \begin{cases} N^{-(\gamma+\alpha-d)/(\alpha+d)}, & \text{if } \gamma - \alpha < d, \\ N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot (\ln N), & \text{if } \gamma - \alpha = d, \\ N^{-2(\gamma+\alpha-d)/(\gamma+\alpha+d)}, & \text{if } \gamma - \alpha > d, \end{cases}$$

which finishes the proof. □

Now, by comparing the Propositions 3.4.1 and 3.4.2, we obtain the theorem.

Proof of Theorem 3.2.1

The theorem is proved by combining the Propositions 3.4.1 and 3.4.2. □

Next, we prove the Theorems 3.3.1 and 3.3.2. To this end, we state the following proposition about the cost of the approximations constructed in Section 3.3.

Proposition 3.4.3

$$\widehat{X}_N^{\#}(T) \in \mathfrak{X}_{c \cdot N}^{\#} \quad \textit{and} \quad \widehat{X}_N^{\mathrm{uni}}(T) \in \mathfrak{X}_{c \cdot N}^{\mathrm{uni}}$$

for some constant $c > 0$, that only depends on the fixed parameters d, α, β, γ and P_μ.

Proof of Proposition 3.4.3

We proof that the constructed algorithms belong to the respective classes. Note, that $\eta = \beta$, if $\gamma \geq \beta \cdot d$. Using (3.46), (3.48), (3.49) and (3.51), we have

$$\operatorname{cost}\left(\widehat{X}_N^{\mathrm{uni}}(T)\right) \leq n^{\mathrm{uni}} \cdot \left|\mathcal{I}_N^{\mathrm{uni}}\right| \preceq N^{P_n + d \cdot P_{\mathcal{I}}} = N.$$

Furthermore, using the parameters defined by (3.56) up to (3.65), we have

$$\begin{aligned}
\operatorname{cost}\left(\widehat{X}_N^{\#}(T)\right) &\leq \sum_{j \in \mathcal{J}_N^{\#}} \nu_j \cdot \left|\mathcal{I}_N^{\#}\right| \\
&\preceq N^{d \cdot P_{\mathcal{I}} + P_\nu} \cdot \begin{cases} \sum\limits_{j \in \mathcal{J}_N^{\#}} (\lambda_j/\mu_j)^{P_\mu}, & \text{if } \eta + \alpha \neq 3d, \\ \sum\limits_{j \in \mathcal{J}_N^{\#}} (\lambda_j/\mu_j)^{P_\mu} / \ln N, & \text{if } \eta + \alpha = 3d. \end{cases}
\end{aligned}$$

By $\mathcal{J}_N^{\#} \subset \{j \in \mathbb{N}^d \mid |j|_2 \leq d \cdot N^{P_{\mathcal{J}}}\}$ and Lemma C.0.3, we derive

$$\begin{aligned}
\operatorname{cost}\left(\widehat{X}_N^{\#}(T)\right) &\preceq N^{d \cdot P_{\mathcal{I}} + P_\nu} \cdot \begin{cases} \sum\limits_{|j|_2 \leq d \cdot N^{P_{\mathcal{J}}}} (\lambda_j/\mu_j)^{P_\mu}, & \text{if } \eta + \alpha \neq 3d, \\ \sum\limits_{|j|_2 \leq d \cdot N^{P_{\mathcal{J}}}} (\lambda_j/\mu_j)^{P_\mu} / \ln N, & \text{if } \eta + \alpha = 3d, \end{cases} \\
&\preceq N^{d \cdot P_{\mathcal{I}} + P_\nu} \cdot \begin{cases} \int\limits_1^{d \cdot N^{P_{\mathcal{J}}}} x^{-(\gamma+\alpha) \cdot P_\mu + d - 1} \, dx, & \text{if } \eta + \alpha \neq 3d, \\ \int\limits_1^{d \cdot N^{P_{\mathcal{J}}}} x^{-(\gamma+\alpha) \cdot P_\mu + d - 1} \, dx / \ln N, & \text{if } \eta + \alpha = 3d, \end{cases} \\
&\preceq N^{d \cdot P_{\mathcal{I}} + P_\nu} \cdot \begin{cases} \left(N^{P_{\mathcal{J}}}\right)^{-(\gamma+\alpha) \cdot P_\mu + d}, & \text{if } \eta + \alpha < 3d, \\ \ln N / \ln N, & \text{if } \eta + \alpha = 3d, \\ 1, & \text{if } \eta + \alpha > 3d, \end{cases} \\
&= N,
\end{aligned}$$

which finishes the proof. □

Now, we give results about the upper error bounds of the constructed algorithms. First, we study the approximation $\widehat{X}_N^{\#}(T)$.

Proposition 3.4.4 *Suppose that*

$$\langle \xi, h_j \rangle^2 \preceq \begin{cases} \lambda_j + \prod_{\ell=1}^{d} j_\ell^{-\gamma/d}, & \text{if } \gamma < \beta \cdot d, \\ \prod_{\ell=1}^{d} j_\ell^{-\beta}, & \text{if } \gamma \geq \beta \cdot d, \end{cases} \tag{3.118}$$

for every $j \in \mathbb{N}^d$. Then for $\gamma \geq \beta \cdot d$ it holds

$$\mathrm{e}\left(\widehat{X}_N^{\#}(T)\right) \preceq \begin{cases} N^{-P_1} \cdot (\ln N)^{(d-1)/2}, & \text{if } \beta + \alpha < 3d, \\ N^{-P_2} \cdot (\ln N)^{\max((d-1)/2, 3/2)}, & \text{if } \beta + \alpha = 3d, \\ N^{-P_3} \cdot (\ln N)^{(d-1)/2}, & \text{if } \beta + \alpha > 3d, \end{cases} \tag{3.119}$$

and for $\gamma < \beta \cdot d$ it holds

$$\mathrm{e}\left(\widehat{X}_N^{\#}(T)\right) \preceq \begin{cases} N^{-P_4} \cdot (\ln N)^{(d-1)/2}, & \text{if } \eta + \alpha < 3d, \\ N^{-P_5} \cdot (\ln N)^{\max((d-1)/2, 3/2)}, & \text{if } \eta + \alpha = 3d, \\ N^{-P_6} \cdot (\ln N)^{(d-1)/2}, & \text{if } \eta + \alpha > 3d, \end{cases} \tag{3.120}$$

with

$$P_1 = \frac{(\gamma + \zeta - d)((\beta - 1)d + \alpha)}{2d((\beta - 1)d + \alpha) + (\gamma + \zeta - d)((3d - \alpha - 1)d + \alpha)},$$
$$P_4 = \frac{(\gamma + \zeta - d)(\gamma + \alpha - d)}{2d(\gamma + \alpha - d) + (\gamma + \zeta - d)((3d - \alpha - 1)d + \alpha + \gamma - \eta d)},$$

and

$$P_2 = P_3 = P_5 = P_6 = \frac{\gamma + \zeta - d}{\gamma + \zeta + d}.$$

In the important case that $d = 1$, the conclusions in Proposition 3.4.4 reduces to

$$\mathrm{e}\left(\widehat{X}_N^{\#}(T)\right) \preceq \begin{cases} N^{-(\gamma+\zeta-1)(\eta+\alpha-1)/(2(\gamma+\zeta+\eta+\alpha-2))}, & \text{if } \eta + \alpha < 3, \\ N^{-(\gamma+\zeta-1)/(\gamma+\zeta+1)} \cdot (\ln N)^{3/2}, & \text{if } \eta + \alpha = 3, \\ N^{-(\gamma+\zeta-1)/(\gamma+\zeta+1)}, & \text{if } \eta + \alpha > 3, \end{cases} \tag{3.121}$$

in the (TC) case and

$$\mathrm{e}\left(\widehat{X}_N^{\#}(T)\right) \preceq \begin{cases} N^{-(\zeta-1)(\alpha-1)/(2(\zeta+\alpha-2))}, & \text{if } \alpha < 3, \\ N^{-(\zeta-1)/(\zeta+1)} \cdot (\ln N)^{3/2}, & \text{if } \alpha = 3, \\ N^{-(\zeta-1)/(\zeta+1)}, & \text{if } \alpha > 3, \end{cases} \tag{3.122}$$

in the (ID) case.

Proof of Proposition 3.4.4

We derive the stated errors of the approximation $\widehat{X}_N^{\#}(T)$. For any algorithm $\widehat{X}_N(T) \in \mathfrak{X}_N^*$ of the form (3.13) approximating the solution (3.9), the Parseval equality and the continuity of the scalar product yields

$$\begin{aligned}
e^2\left(\widehat{X}_N(T)\right) &= \mathrm{E}\left\|X(T) - \widehat{X}_N(T)\right\|^2 \\
&= \mathrm{E}\Bigg\|\sum_{j\in\mathbb{N}^d}\left(\exp(-\mu_j T)\cdot\langle\xi, h_j\rangle + \sum_{i\in\mathbb{N}^d}\lambda_i^{1/2}\cdot Z_{ij}(T)\right)\cdot h_j \\
&\qquad - \sum_{j\in\mathcal{J}_N}\left(\exp(-\mu_j T)\cdot\langle\xi, h_j\rangle + \sum_{i\in\mathcal{I}_N}\lambda_i^{1/2}\cdot \widehat{Z}_{ij,N}(T)\right)\cdot h_j\Bigg\|^2 \\
&= \mathrm{E}\sum_{k\in\mathbb{N}^d}\Bigg(\sum_{j\notin\mathcal{J}_N}\left(\exp(-\mu_j T)\cdot\langle\xi, h_j\rangle + \sum_{i\in\mathbb{N}^d}\lambda_i^{1/2}\cdot Z_{ij,N}(T)\right)\cdot\langle h_j, h_k\rangle \\
&\qquad + \sum_{j\in\mathcal{J}_N}\sum_{i\in\mathcal{I}_N}\lambda_i^{1/2}\cdot\left(Z_{ij}(T) - \widehat{Z}_{ij,N}(T)\right)\cdot\langle h_j, h_k\rangle \\
&\qquad + \sum_{j\in\mathcal{J}_N}\sum_{i\notin\mathcal{I}_N}\lambda_i^{1/2}\cdot Z_{ij,N}(T)\cdot\langle h_j, h_k\rangle\Bigg)^2 .
\end{aligned}$$

The drift-implicit Euler-Maruyama scheme (3.42) implies

$$\widehat{Z}_{ij,N}(T) = \sum_{k=0}^{n_i-1} B_{ij}(t_{k,i})\cdot\Delta_{k,i}\beta_i\prod_{\ell=k}^{n_i-1}\left(1+\mu_j\cdot\Delta_{\ell,i}\right)^{-1} \tag{3.123}$$

and inserting uniform time nodes in (3.123), gives

$$\widehat{Z}_{ij,N}^{\mathrm{uni}}(T) = \sum_{k=0}^{n-1} B_{ij}(\frac{k}{n}T)\cdot\left(\beta_i(\frac{k+1}{n}T) - \beta_i(\frac{k}{n}T)\right)\prod_{\ell=k}^{n-1}\left(1+\mu_j\cdot\frac{1}{n}T\right)^{-1}, \tag{3.124}$$

with $i, j \in \mathbb{N}^d$. We know that $E(Z_{ij}(T)) = 0$ for every $i, j \in \mathbb{N}^d$ and $\mathrm{E}(\widehat{Z}_{ij,N}(T)) = 0$ for every $i \in \mathcal{I}_N$ and $j \in \mathcal{J}_N$ as well as $(\beta_i)_{i\in\mathbb{N}^d}$ is an independent family of scalar

Brownian motions and $\langle h_i, h_k\rangle \cdot \langle h_j, h_k\rangle = 0$ for every $k \in \mathbb{N}^d$ if $i \neq j$. Therefore, we conclude for those approximations using the Euler-Maruyama schemes

$$\begin{aligned}
\mathrm{e}^2\left(\widehat{X}_N(T)\right) &= \sum_{j\notin\mathcal{J}_N} \exp(-2\mu_j T)\cdot \langle \xi, h_j\rangle^2 \\
&\quad + \sum_{j\in\mathcal{J}_N}\sum_{i\in\mathcal{I}_N} \lambda_i \cdot \mathrm{E}\left(Z_{ij}(T) - \widehat{Z}_{ij,N}(T)\right)^2 \\
&\quad + \sum_{j\in\mathcal{J}_N}\sum_{i\notin\mathcal{I}_N} \lambda_i \cdot \mathrm{E}Z_{ij}^2(T) + \sum_{j\notin\mathcal{J}_N}\sum_{i\in\mathbb{N}^d} \lambda_i \cdot \mathrm{E}Z_{ij}^2(T) \qquad (3.125)
\end{aligned}$$

We estimate an upper bound for the summand of the first series in (3.125) by

$$\exp(-2\mu_j T)\cdot\langle\xi, h_j\rangle^2 \preceq \frac{1}{\mu_j}\cdot\begin{cases}\lambda_j + \prod_{\ell=1}^d j_\ell^{-\gamma/d}, & \text{if } \gamma < \beta\cdot d,\\ \prod_{\ell=1}^d j_\ell^{-\beta}, & \text{if } \gamma \geq \beta\cdot d,\end{cases} \qquad (3.126)$$

using $\exp(-x) < 1/x$ for $x > 0$ and (3.118). From (3.10), the Ito isometry and (3.4), we get

$$\begin{aligned}
\mathrm{E}Z_{ij}^2(T) &= \int_0^T \exp(-2\mu_j(T-t))\cdot (B_{ij}(t))^2\, dt \\
&\preceq \frac{1}{\mu_j}\cdot\begin{cases}\prod_{\substack{\ell=1\\ i_\ell\neq j_\ell}}^d |i_\ell - j_\ell|^{-\beta}, & \text{if } i\neq j,\\ 1, & \text{if } i = j,\end{cases}
\end{aligned}$$

for $i, j\in\mathbb{N}^d$. Thus, we obtain by Lemma C.0.4,

$$\sum_{j\in\mathcal{J}_N}\sum_{i\notin\mathcal{I}_N}\lambda_i\cdot\mathrm{E}Z_{ij}^2(T) \preceq \sum_{i\notin\mathcal{I}_N} |i|_2^{-(\gamma+\zeta)}. \qquad (3.127)$$

On the other hand, using Lemma C.0.5 yields

$$\sum_{j\notin\mathcal{J}_N}\sum_{i\in\mathbb{N}^d}\lambda_i\cdot\mathrm{E}Z_{ij}^2(T) \preceq \begin{cases}\sum_{j\notin\mathcal{J}_N}\mu_j^{-1}\cdot\left(\lambda_j + \prod_{\ell=1}^d j_\ell^{-\gamma/d}\right), & \text{if } \gamma<\beta\cdot d,\\ \sum_{j\notin\mathcal{J}_N}\mu_j^{-1}\cdot\prod_{\ell=1}^d j_\ell^{-\beta}, & \text{if } \gamma\geq\beta\cdot d.\end{cases} \qquad (3.128)$$

Inserting (3.126), (3.127) and (3.128) in (3.125), we have for every $\widehat{X}_N(T) \in \mathfrak{X}_N^*$ that uses the Euler-Maruyama schemes by the assumptions of the proposition

$$
\begin{aligned}
\mathrm{e}^2\left(\widehat{X}_N(T)\right) \preceq\ & \sum_{j\in\mathcal{J}_N}\sum_{i\in\mathcal{I}_N} |i|_2^{-\gamma}\cdot \mathrm{E}\left(Z_{ij}(T)-\widehat{Z}_{ij,N}(T)\right)^2 \\
&+\sum_{i\notin\mathcal{I}_N} |i|_2^{-(\gamma+\zeta)} \\
&+\begin{cases} \sum\limits_{j\notin\mathcal{J}_N} |j|_2^{-(\gamma+\alpha)} + \sum\limits_{j\notin\mathcal{J}_N} |j|_2^{-\alpha}\prod\limits_{\ell=1}^{d} j_\ell^{-\gamma/d}, & \text{if } \gamma<\beta\cdot d,\\ \sum\limits_{j\notin\mathcal{J}_N} |j|_2^{-\alpha}\prod\limits_{\ell=1}^{d} j_\ell^{-\beta}, & \text{if } \gamma\ge\beta\cdot d. \end{cases}
\end{aligned}
\tag{3.129}
$$

To derive an upper error bound of the approximation $\widehat{X}_N^{\#}(T)$, we use the algorithm $\widehat{Z}_{ij,N}^{\#}(T)$ defined by (3.55) as the approximation scheme $\widehat{Z}_{ij,N}(T)$ in (3.129). Put

$$\Delta s_{k,j} = s_{k+1,j} - s_{k,j}$$

and

$$\Pi s_{k,j} = \prod_{\ell=k}^{\nu_j-1} (1+\mu_j\cdot\Delta s_{k,j})^{-1}$$

for every $k \in \{0, \ldots, \nu_j - 1\}$ and $j \in \mathcal{J}_N^{\#}$. Then, the Ito isometry yields for (3.39) and (3.55) with $i \in \mathcal{I}_N^{\#}$ and $j \in \mathcal{J}_N^{\#}$,

$$
\begin{aligned}
& \mathrm{E}\left(Z_{ij}(T) - \widehat{Z}_{ij,N}^{\#}(T)\right)^2 \\
= \; & \mathrm{E}\left(\sum_{k=0}^{\nu_j-1} \int_{s_{k,j}}^{s_{k+1,j}} \left(\exp(-\mu_j(T-s)) \cdot B_{ij}(s) - \Pi s_{k,j} \cdot B_{ij}(s_{k,j})\right) d\beta_i(s)\right)^2 \\
= \; & \sum_{k=0}^{\nu_j-1} \int_{s_{k,j}}^{s_{k+1,j}} \left(\exp(-\mu_j(T-s)) \cdot B_{ij}(s) - \Pi s_{k,j} \cdot B_{ij}(s_{k,j})\right)^2 \, ds \\
\leq \; & 4\left(\sum_{k=0}^{\nu_j-1} \int_{s_{k,j}}^{s_{k+1,j}} \left(\exp(-\mu_j(T-s)) \cdot B_{ij}(s) - \exp(-\mu_j(T-s_{k,j})) \cdot B_{ij}(s)\right)^2 \, ds \right. \\
& + \sum_{k=0}^{\nu_j-1} \int_{s_{k,j}}^{s_{k+1,j}} \left(\exp(-\mu_j(T-s_{k,j})) \cdot B_{ij}(s) - \exp(-\mu_j(T-s_{k,j})) \cdot B_{ij}(s_{k,j})\right)^2 \, ds \\
& \left. + \sum_{k=0}^{\nu_j-1} \int_{s_{k,j}}^{s_{k+1,j}} \left(\exp(-\mu_j(T-s_{k,j})) \cdot B_{ij}(s_{k,j}) - \Pi s_{k,j} \cdot B_{ij}(s_{k,j})\right)^2 \, ds \right).
\end{aligned}
$$

Thus, by (3.4), Lemma C.0.7 and the mean value theorem, we obtain for $i \neq j$,

$$
\begin{aligned}
&\mathrm{E}\left(Z_{ij}(T) - \widehat{Z}^{\#}_{ij,N}(T)\right)^2 \\
&\preceq \left(\prod_{\substack{\ell=1\\ i_\ell \neq j_\ell}}^{d} |i_\ell - j_\ell|^{-\beta}\right) \cdot \sum_{k=0}^{\nu_j-1} \int_{s_{k,j}}^{s_{k+1,j}} (\exp(-\mu_j(T-s)) - \exp(-\mu_j(T-s_{k,j})))^2 \, ds \\
&\quad + \sum_{k=0}^{\nu_j-1} \exp(-2\mu_j(T-s_{k,j})) \int_{s_{k,j}}^{s_{k+1,j}} (B_{ij}(s) - B_{ij}(s_{k,j}))^2 \, ds \\
&\quad + \left(\prod_{\substack{\ell=1\\ i_\ell \neq j_\ell}}^{d} |i_\ell - j_\ell|^{-\beta}\right) \cdot \sum_{k=0}^{\nu_j-1} \int_{s_{k,j}}^{s_{k+1,j}} \left(\exp(-\mu_j(T-s_{k,j})) - \Pi s_{k,j}\right)^2 \, ds \\
&\preceq \left(\prod_{\substack{\ell=1\\ i_\ell \neq j_\ell}}^{d} |i_\ell - j_\ell|^{-\beta}\right) \cdot \frac{1}{\mu_j \nu_j^2} \\
&\quad + \left(\prod_{\substack{\ell=1\\ i_\ell \neq j_\ell}}^{d} |i_\ell - j_\ell|^{-\beta}\right) \cdot \sum_{k=0}^{\nu_j-1} \exp(-2\mu_j(T-s_{k,j}))(\Delta s_{k,j})^3.
\end{aligned}
$$

We observe, that

$$
\int_{s_{k,j}}^{s_{k+1,j}} \exp(-\frac{\mu_j}{3}(T-t)) \, dt \geq \Delta s_{k,j} \cdot \exp(-\frac{\mu_j}{3}(T-s_{k,j}))
$$

for every $k \in \{0, \ldots, \nu_j - 1\}$ and therefore

$$
\Delta s_{k,j} \leq \frac{1}{\nu_j} \exp(\frac{\mu_j}{3}(T-s_{k,j})) \int_0^T \exp(-\frac{\mu_j}{3}(T-t)) \, dt \leq \frac{3}{\mu_j \nu_j} \exp(\frac{\mu_j}{3}(T-s_{k,j})).
$$

Hence, we get

$$\begin{aligned}
\mathrm{E}\left(Z_{ij}(T)-\widehat{Z}^{\#}_{ij,N}(T)\right)^2 &\preceq \left(\prod_{\substack{\ell=1\\ i_\ell\neq j_\ell}}^{d}|i_\ell-j_\ell|^{-\beta}\right)\cdot\frac{1}{\mu_j\nu_j^2}\\
&\quad+\left(\prod_{\substack{\ell=1\\ i_\ell\neq j_\ell}}^{d}|i_\ell-j_\ell|^{-\beta}\right)\cdot\frac{1}{\mu_j^3\nu_j^3}\sum_{k=0}^{\nu_j-1}\exp(-\mu_j(T-s_{k,j}))\\
&\preceq \left(\prod_{\substack{\ell=1\\ i_\ell\neq j_\ell}}^{d}|i_\ell-j_\ell|^{-\beta}\right)\cdot\left(\frac{1}{\mu_j\nu_j^2}+\frac{1}{\mu_j^3\nu_j^2}\right)\\
&\preceq \left(\prod_{\substack{\ell=1\\ i_\ell\neq j_\ell}}^{d}|i_\ell-j_\ell|^{-\beta}\right)\cdot\frac{1}{\mu_j\nu_j^2}. \qquad (3.130)
\end{aligned}$$

For $i=j$ the analog calculation yields

$$\mathrm{E}\left(Z_{ii}(T)-\widehat{Z}^{\#}_{ii,N}(T)\right)^2\preceq\frac{1}{\mu_i\nu_i^2}. \qquad (3.131)$$

Note that

$$|j|_2^{-\gamma}\leq\prod_{\ell=1}^{d}j_\ell^{-\gamma/d} \qquad (3.132)$$

for every $j\in\mathbb{N}^d$, if $d\in\mathbb{N}$ and $\gamma\in\{x\in\mathbb{R}\,|\,x>d\}\cup\{0\}$. Moreover,

$$\sum_{j\notin\mathcal{J}_N^{\#}}\prod_{\ell=1}^{d}j_\ell^{-(\gamma+\alpha)/d}\preceq(N^{P_{\mathcal{J}}})^{-(\gamma+\alpha)/d+1}\cdot(\ln N)^{d-1}, \qquad (3.133)$$

if $\gamma<\beta\cdot d$ and

$$\sum_{j\notin\mathcal{J}_N^{\#}}\prod_{\ell=1}^{d}j_\ell^{-\beta-\alpha/d}\preceq(N^{P_{\mathcal{J}}})^{-\beta-\alpha/d+1}\cdot(\ln N)^{d-1}, \qquad (3.134)$$

if $\gamma\geq\beta\cdot d$, which follows from [PW90], Section 2.2. Now, we apply the assumptions of the proposition as well as (3.53), (3.54), (3.58), (3.63), and (3.130) up to (3.134) in

(3.129) to estimate in the case $\eta + \alpha \neq 3d$

$$\begin{aligned}
\mathrm{e}^2\left(\widehat{X}_N^{\#}(T)\right) \;\preceq\;& N^{-2P_\nu} \cdot \sum_{|j|_2 \le d \cdot N^{P_\mathcal{J}}} \left(|j|_2^{\gamma}\right)^{2P_\mu} \cdot \left(|j|_2^{\alpha}\right)^{2P_\mu - 1} \sum_{\substack{|i|_2 \le N^{P_\mathcal{I}} \\ i \neq j}} |i|_2^{-\gamma} \cdot \prod_{\substack{\ell=1 \\ i_\ell \neq j_\ell}}^{d} |i_\ell - j_\ell|^{-\beta} \\
&+ N^{-2P_\nu} \cdot \sum_{|i|_2 \le d \cdot N^{P_\mathcal{J}}} |i|_2^{(\gamma+\alpha)\cdot(2P_\mu - 1)} + \sum_{|i|_2 > N^{P_\mathcal{I}}} |i|_2^{-(\gamma+\zeta)} \\
&+ (\ln N)^{d-1} \cdot \begin{cases} (N^{P_\mathcal{J}})^{-(\gamma+\alpha)/d+1}, & \text{if } \gamma < \beta \cdot d, \\ (N^{P_\mathcal{J}})^{-\beta-\alpha/d-1}, & \text{if } \gamma \ge \beta \cdot d. \end{cases}
\end{aligned}$$

Thus, by Lemma C.0.3 and C.0.4, we get

$$\begin{aligned}
\mathrm{e}^2\left(\widehat{X}_N^{\#}(T)\right) \;\preceq\;& N^{-2P_\nu} \cdot \sum_{|j|_2 \le d \cdot N^{P_\mathcal{J}}} |j|_2^{2P_\mu \cdot (\gamma+\alpha) - \eta - \alpha} \\
&+ \sum_{|i|_2 > N^{P_\mathcal{I}}} |i|_2^{-(\gamma+\zeta)} \\
&+ (\ln N)^{d-1} \cdot \begin{cases} (N^{P_\mathcal{J}})^{-(\gamma+\alpha)/d+1}, & \text{if } \gamma < \beta \cdot d, \\ (N^{P_\mathcal{J}})^{-\beta-\alpha/d-1}, & \text{if } \gamma \ge \beta \cdot d, \end{cases} \\
\preceq\;& N^{-2P_\nu} \cdot \int_1^{d \cdot N^{P_\mathcal{J}}} x^{2P_\mu \cdot (\gamma+\alpha) - \eta - \alpha + d - 1}\, dx \\
&+ \int_{N^{P_\mathcal{I}}}^{\infty} x^{-(\gamma+\zeta)+d-1}\, dx \\
&+ (\ln N)^{d-1} \cdot \begin{cases} (N^{P_\mathcal{J}})^{-(\gamma+\alpha)/d+1}, & \text{if } \gamma < \beta \cdot d, \\ (N^{P_\mathcal{J}})^{-\beta-\alpha/d-1}, & \text{if } \gamma \ge \beta \cdot d, \end{cases} \\
\preceq\;& N^{-2P_\nu} \cdot \begin{cases} (N^{P_\mathcal{J}})^{2P_\mu \cdot (\gamma+\alpha) - \eta - \alpha + d}, & \text{if } P_\mu > (\eta + \alpha - d)/2(\gamma+\alpha), \\ \ln N, & \text{if } P_\mu = (\eta + \alpha - d)/2(\gamma+\alpha), \\ 1, & \text{if } P_\mu < (\eta + \alpha - d)/2(\gamma+\alpha), \end{cases} \\
&+ (N^{P_\mathcal{I}})^{-(\gamma+\zeta-d)} \\
&+ (\ln N)^{d-1} \cdot \begin{cases} (N^{P_\mathcal{J}})^{-(\gamma+\alpha)/d+1}, & \text{if } \gamma < \beta \cdot d, \\ (N^{P_\mathcal{J}})^{-\beta-\alpha/d-1}, & \text{if } \gamma \ge \beta \cdot d. \end{cases}
\end{aligned}$$

Hence, with the parameters chosen from (3.56) to (3.65) the proposition is proven in the case $\eta + \alpha \neq 3d$. In the case $\eta + \alpha = 3d$, we estimate the error in the same way as above, but from (3.58) and (3.63) as well as by integration, we obtain an additional factor $(\ln N)^3$, which finishes the proof. □

Next, we turn to $\widehat{X}_N^{\text{uni}}(T)$. For this approximation, we distinguish the cases $d = 1$ and $d \in \mathbb{N} \setminus \{1\}$ in the following two propositions. For our results in the case $d \in \mathbb{N} \setminus \{1\}$ we have furthermore to consider $\alpha \leq d$, which definitely covers the special important value $\alpha = 2$, yet.

Proposition 3.4.5 *Let $d = 1$ and suppose that*

$$\langle \xi, h_j \rangle^2 \preceq \begin{cases} j^{-\gamma}, & \textit{if} \quad \gamma \leq \beta, \\ j^{-\beta}, & \textit{if} \quad \gamma > \beta, \end{cases} \tag{3.135}$$

for every $j \in \mathbb{N}$. Then

$$\mathrm{e}\left(\widehat{X}_N^{\text{uni}}(T)\right) \preceq \begin{cases} N^{-(\gamma+\zeta-1)(\alpha+\eta-1)/(2(\alpha(\gamma+\zeta)+\eta-1))}, & \textit{if} \quad \eta - \alpha < 1, \\ N^{-(\gamma+\zeta-1)/(\gamma+\zeta+1)} \cdot (\ln N)^{1/2}, & \textit{if} \quad \eta - \alpha = 1, \\ N^{-(\gamma+\zeta-1)/(\gamma+\zeta+1)}, & \textit{if} \quad \eta - \alpha > 1, \end{cases} \tag{3.136}$$

in the (TC) case and

$$\mathrm{e}\left(\widehat{X}_N^{\text{uni}}(T)\right) \preceq \begin{cases} N^{-(\alpha-1)/(2(\alpha+1))}, & \textit{if} \quad \alpha \leq \beta, \\ N^{-(\alpha-1)(\beta-1)/(2(\alpha\beta-1))}, & \textit{if} \quad \alpha > \beta, \end{cases} \tag{3.137}$$

in the (ID) case.

Proof of Proposition 3.4.5

We derive the upper error bounds of the approximation $\widehat{X}_N^{\text{uni}}(T)$ in the case $d = 1$. First, consider in general $d \in \mathbb{N}$ and all the assumptions used in the Propositions 3.4.3 and 3.4.4. Now, we use the scheme $\widehat{Z}_{ij,N}^{\text{uni}}(T)$ given by (3.124) in (3.129). Thus, we obtain for $\widehat{X}_N^{\mathrm{uni}}(T)$,

$$\mathrm{e}^2\left(\widehat{X}_N^{\mathrm{uni}}(T)\right) \preceq \sum_{j\in\mathcal{J}_N}\sum_{i\in\mathcal{I}_N} |i|_2^{-\gamma}\cdot \mathrm{E}\left(Z_{ij}(T)-\widehat{Z}_{ij,N}^{\mathrm{uni}}(T)\right)^2 + \sum_{i\notin\mathcal{I}_N}|i|_2^{-(\gamma+\zeta)} + \begin{cases} \sum_{j\notin\mathcal{J}_N}|j|_2^{-(\gamma+\alpha)} + \sum_{j\notin\mathcal{J}_N}|j|_2^{-\alpha}\prod_{\ell=1}^d j_\ell^{-\gamma/d}, & \text{if } \gamma<\beta\cdot d,\\ \sum_{j\notin\mathcal{J}_N}|j|_2^{-\alpha}\prod_{\ell=1}^d j_\ell^{-\beta}, & \text{if } \gamma\geq\beta\cdot d. \end{cases} \tag{3.138}$$

Remember (3.123) and put for notational convienience

$$\Pi_{k,ij} = \prod_{\ell=k}^{n_i-1}(1+\mu_j\cdot\Delta_{\ell,i})^{-1}$$

for $i,j\in\mathbb{N}^d$ and $k=0,\dots,n_i-1$. Note, that for uniform time nodes holds

$$\Pi_{k,ij} = \prod_{\ell=k}^{n-1}\left(1+\mu_j\cdot\frac{1}{n}T\right)^{-1} = \left(1+\mu_j\cdot\frac{1}{n}T\right)^{-(n-k)}.$$

Consequently, by (3.39), (3.124) and the Ito isometry, we obtain for $i\in\mathcal{I}_N^{\mathrm{uni}}$, $j\in\mathcal{J}_N^{\mathrm{uni}}$ and $n=n^{\mathrm{uni}}$,

$$\begin{aligned}
&\mathrm{E}\left(Z_{ij}(T)-\widehat{Z}_{ij,N}^{\mathrm{uni}}(T)\right)^2\\
&= \mathrm{E}\left(\sum_{k=0}^{n-1}\int_{\frac{k}{n}T}^{\frac{k+1}{n}T}\left(\exp(-\mu_j(T-t))\cdot B_{ij}(t)-\Pi_{k,ij}\cdot B_{ij}(\frac{k}{n}T)\right)d\beta_i(t)\right)^2\\
&= \sum_{k=0}^{n-1}\int_{\frac{k}{n}T}^{\frac{k+1}{n}T}\left(\exp(-\mu_j(T-t))\cdot B_{ij}(t)-\left(1+\mu_j\cdot\frac{1}{n}T\right)^{-(n-k)}\cdot B_{ij}(\frac{k}{n}T)\right)^2 dt\\
&\leq 2\left(\sum_{k=0}^{n-1}\int_{\frac{k}{n}T}^{\frac{k+1}{n}T}\left(\exp(-\mu_j(T-t))\cdot B_{ij}(t)-\exp(-\mu_j(T-t))\cdot B_{ij}(\frac{k}{n}T)\right)^2 dt \right. \qquad (3.139)\\
&\quad\left.+\sum_{k=0}^{n-1}\int_{\frac{k}{n}T}^{\frac{k+1}{n}T}\left(\exp(-\mu_j(T-t))\cdot B_{ij}(\frac{k}{n}T)-\left(1+\mu_j\cdot\frac{1}{n}T\right)^{-(n-k)}\cdot B_{ij}(\frac{k}{n}T)\right)^2 dt\right).
\end{aligned}$$

The parameters in (3.47) and (3.48) are chosen in a way, such that $n \succeq \mu_j$ holds for every $j \in \mathcal{J}_N^{\text{uni}}$. Therefore, let $d = 1$ and assume without a loss of generality $n \geq \max(\mu_j, T)$ for every $j \in \mathcal{J}_N^{\text{uni}}$, which means that N is sufficiently large. Thus, by (3.4), the mean value theorem and Lemma C.0.6, we get

$$
\begin{aligned}
&\mathrm{E}\left(Z_{ij}(T) - \widehat{Z}_{ij,N}^{\text{uni}}(T)\right)^2 \\
&\preceq \sum_{k=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \exp(-2\mu_j(T-t)) \cdot \left(B_{ij}(t) - B_{ij}(\frac{k}{n}T)\right)^2 \, dt \\
&\quad + \left(|i-j|^\beta + 1\right)^{-1} \cdot \sum_{k=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \left(\exp(-\mu_j(T-t)) - \left(1 + \mu_j \cdot \frac{1}{n}T\right)^{-(n-k)}\right)^2 \, dt \\
&\preceq \left(|i-j|^\beta + 1\right)^{-1} \cdot \sum_{k=0}^{n-1} \exp\left(-2\mu_j(T - \frac{k+1}{n}T)\right) \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \left(t - \frac{k}{n}T\right)^2 \, dt \\
&\quad + \left(|i-j|^\beta + 1\right)^{-1} \cdot \frac{\mu_j}{n^2} \\
&= \left(|i-j|^\beta + 1\right)^{-1} \cdot \frac{T^3}{n^3} \sum_{k=0}^{n-1} \exp\left(-2\mu_j(T - \frac{k+1}{n}T)\right) \\
&\quad + \left(|i-j|^\beta + 1\right)^{-1} \cdot \frac{\mu_j}{n^2} \\
&\leq \left(|i-j|^\beta + 1\right)^{-1} \cdot \frac{T^3}{n^2} + \left(|i-j|^\beta + 1\right)^{-1} \cdot \frac{\mu_j}{n^2} \\
&\preceq \left(|i-j|^\beta + 1\right)^{-1} \cdot \frac{\mu_j}{n^2}.
\end{aligned}
\tag{3.140}
$$

Hence, inserting (3.2), (3.5), (3.43), (3.44), (3.45), (3.135) and (3.140) in (3.138), we conclude by Lemma C.0.4

$$
\begin{aligned}
\mathrm{e}^2\left(\widehat{X}_N^{\text{uni}}(T)\right) &\preceq N^{-2P_n} \cdot \sum_{j \leq N^{P_{\mathcal{J}}}} \sum_{i \leq N^{P_{\mathcal{I}}}} \left(|i-j|^\beta + 1\right)^{-1} \cdot i^{-\gamma} \cdot j^\alpha \\
&\quad + \sum_{i > N^{P_{\mathcal{I}}}} i^{-(\gamma+\zeta)} + \sum_{j > N^{P_{\mathcal{J}}}} j^{-(\eta+\alpha)} \\
&\preceq N^{-2P_n} \cdot \sum_{j \leq N^{P_{\mathcal{J}}}} j^{-\eta+\alpha} + \sum_{i > N^{P_{\mathcal{I}}}} i^{-(\gamma+\zeta)} + \sum_{j > N^{P_{\mathcal{J}}}} j^{-(\eta+\alpha)}.
\end{aligned}
$$

Finally, we use Lemma C.0.3 to estimate

$$
\begin{aligned}
\mathrm{e}^2\left(\widehat{X}_N^{\mathrm{uni}}(T)\right) &\preceq N^{-2P_n} \cdot \int_1^{N^{P_{\mathcal{J}}}} x^{-\eta+\alpha}\,dx + \int_{N^{P_{\mathcal{I}}}}^{\infty} x^{-(\gamma+\zeta)}\,dx + \int_{N^{P_{\mathcal{J}}}}^{\infty} x^{-(\eta+\alpha)}\,dx \\
&\preceq N^{-2P_n} \cdot \begin{cases} (N^{P_{\mathcal{J}}})^{-\eta+\alpha+1}, & \text{if } \eta-\alpha<1, \\ \ln N, & \text{if } \eta-\alpha=1, \\ 1, & \text{if } \eta-\alpha>1, \end{cases} \\
&\quad +(N^{P_{\mathcal{I}}})^{-\gamma-\zeta+1} + (N^{P_{\mathcal{J}}})^{-\eta-\alpha+1}.
\end{aligned}
$$

Applying (3.46), (3.47) and (3.48) finishes the proof. □

Proposition 3.4.6 *Let $d \in \mathbb{N}\backslash\{1\}$ and $\alpha \leq d$. Furthermore, suppose (3.118) for every $j \in \mathbb{N}^d$. Then for $\gamma \geq \beta \cdot d$ it holds*

$$
\mathrm{e}\left(\widehat{X}_N^{\mathrm{uni}}(T)\right) \preceq \begin{cases} N^{-R_1} \cdot (\ln N)^{(d-1)/2}, & \text{if} \quad \beta-\alpha<d, \\ N^{-R_2} \cdot (\ln N)^{d/2}, & \text{if} \quad \beta-\alpha=d, \\ N^{-R_3} \cdot (\ln N)^{(d-1)/2}, & \text{if} \quad \beta-\alpha>d, \end{cases} \tag{3.141}
$$

with

$$
R_1 = \frac{(\gamma+\zeta-d)((\beta-1)d+\alpha)}{2d((\beta-1)d+\alpha)+(\gamma+\zeta-d)((d+1)\alpha+d(d-1))},
$$

and

$$
R_2 = R_3 = \frac{\gamma+\zeta-d}{\gamma+\zeta+d}.
$$

Proof of Proposition 3.4.6

Here we derive the upper error bounds of the approximation $\widehat{X}_N^{\mathrm{uni}}(T)$ in the case $d \in \mathbb{N} \setminus \{1\}$. Therefore, we start with the estimation given by (3.138) for any $d \in \mathbb{N}$ and let $n = n^{\mathrm{uni}}$. To find an upper bound for the term $\mathrm{E}(Z_{ij}(T) - \widehat{Z}_{ij,N}^{\mathrm{uni}}(T))^2$, we also use the inequality (3.139) of the previous proof. If $\alpha \leq d \in \mathbb{N} \setminus \{1\}$, we see that $n \succeq \mu_j$ for every $j \in \mathcal{J}_N^{\mathrm{uni}}$ using (3.44) and (3.45) with the chosen parameters (3.50) and (3.51). Thus, without loss of generality assume N sufficiently large, such that $n \geq \max(\mu_j, T)$

for every $j \in \mathcal{J}_N^{\text{uni}}$. Hence, use (3.4) and the analogous estimation as for (3.140) to conclude for $i \neq j$

$$\mathrm{E}\left(Z_{ij}(T) - \widehat{Z}_{ij,N}^{\text{uni}}(T)\right)^2 \preceq \left(\prod_{\substack{\ell=1 \\ i_\ell \neq j_\ell}}^{d} |i_\ell - j_\ell|^{-\beta}\right) \cdot \frac{\mu_j}{n^2} \tag{3.142}$$

and

$$\mathrm{E}\left(Z_{ii}(T) - \widehat{Z}_{ii,N}^{\text{uni}}(T)\right)^2 \preceq \frac{\mu_i}{n^2}. \tag{3.143}$$

Then Lemma C.0.4, together with (3.2), (3.5), (3.43), (3.44), (3.45), (3.118), (3.142) and (3.143) in (3.138) yields

$$\begin{aligned}
\mathrm{e}^2\left(\widehat{X}_N^{\text{uni}}(T)\right) \;&\preceq\; N^{-2P_n} \cdot \sum_{|j|_2 \leq d \cdot N^{P_\mathcal{J}}} \sum_{\substack{|i|_2 \leq N^{P_\mathcal{I}} \\ i \neq j}} \left(\prod_{\substack{\ell=1 \\ i_\ell \neq j_\ell}}^{d} |i_\ell - j_\ell|^{-\beta}\right) \cdot |i|_2^{-\gamma} \cdot |j|_2^{\alpha} \\
&\quad + N^{-2P_n} \cdot \sum_{|j|_2 \leq d \cdot N^{P_\mathcal{J}}} |j|_2^{-\gamma+\alpha} \\
&\quad + \sum_{|i|_2 > N^{P_\mathcal{I}}} |i|_2^{-(\gamma+\zeta)} + \sum_{j \notin \mathcal{J}_N^{\text{uni}}} |j|_2^{-\alpha} \prod_{\ell=1}^{d} j_\ell^{-\beta} \\
&\preceq\; N^{-2P_n} \cdot \sum_{|j|_2 \leq d \cdot N^{P_\mathcal{J}}} |j|_2^{-\eta+\alpha} + \sum_{|i|_2 > N^{P_\mathcal{I}}} |i|_2^{-(\gamma+\zeta)} + \sum_{j \notin \mathcal{J}_N^{\text{uni}}} \prod_{\ell=1}^{d} j_\ell^{-(\beta+\alpha/d)}.
\end{aligned}$$

From Section 2.2 in [PW90] and (3.44), we get

$$\sum_{j \notin \mathcal{J}_N^{\text{uni}}} \prod_{\ell=1}^{d} j_\ell^{-(\beta+\alpha/d)} \preceq (N^{P_\mathcal{J}})^{-(\beta+\alpha/d-1)} \cdot (\ln N)^{d-1}. \tag{3.144}$$

Thus, by Lemma C.0.3 and $\gamma \geq \beta \cdot d$ holds

$$\begin{aligned} \mathrm{e}^2\left(\widehat{X}_N^{\mathrm{uni}}(T)\right) &\preceq N^{-2P_n} \cdot \int_1^{d\cdot N^{P_{\mathcal{J}}}} x^{-\eta+\alpha+d-1}\,dx \\ &\quad + \int_{N^{P_{\mathcal{I}}}} x^{-\gamma-\zeta+d-1}\,dx + (N^{P_{\mathcal{J}}})^{-(\beta+\alpha/d-1)} \cdot (\ln N)^{d-1} \\ &\preceq N^{-2P_n} \cdot \begin{cases} (N^{P_{\mathcal{J}}})^{-\beta+\alpha+d}, & \text{if } \beta-\alpha<d, \\ \ln N, & \text{if } \beta-\alpha=d, \\ 1, & \text{if } \beta-\alpha>1, \end{cases} \\ &\quad +(N^{P_{\mathcal{I}}})^{-(\gamma+\zeta-d)} + (N^{P_{\mathcal{J}}})^{-(\beta+\alpha/d-1)} \cdot (\ln N)^{d-1}. \end{aligned}$$

Using the parameters defined by (3.49), (3.50) and (3.51) completes the proof. □

Finally, we provide lower bounds for the error of every algorithm $\widehat{X}(T) \in \mathfrak{X}_N^{\#}$ and $\widehat{X}(T) \in \mathfrak{X}_N^{\mathrm{uni}}$ approximating the solution (3.38). For this purpose, we consider stochastic evolution equations of the type (3.1) using time-independent operators B in the diffusion term satisfying the conditions of Assumption 3.0.2. Indeed, the processes $(Z_{ij}(t))_{t\in[0,T]}$, with $i,j \in \mathbb{N}^d$, form a coupled system of Ornstein-Uhlenbeck processes and we obtain the following result.

Proposition 3.4.7 *Suppose that $B_{ij} : [0,T] \to \mathbb{R}$ is constant, i.e.*

$$B_{ij} = B_{ij}(t), \quad t \in [0,T], \tag{3.145}$$

for every $i,j \in \mathbb{N}^d$. Then

$$\mathrm{e}_N^{\#} \succeq N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)} \tag{3.146}$$

and

$$\mathrm{e}_N^{\mathrm{uni}} \succeq \begin{cases} N^{-(\gamma+\alpha-d)/(2(\alpha+d))}, & \text{if } \gamma-\alpha<d, \\ N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)} \cdot (\ln N)^{1/2}, & \text{if } \gamma-\alpha=d, \\ N^{-(\gamma+\alpha-d)/(\gamma+\alpha+d)}, & \text{if } \gamma-\alpha>d. \end{cases} \tag{3.147}$$

Proof of Proposition 3.4.7

First, we consider any approximation $\widehat{X}(T) \in \mathfrak{X}_N^*$ of $X(T)$. For the error of such an approximation, we have by (3.38),

$$\begin{aligned} &\mathrm{E}\left\|X(T)-\widehat{X}(T)\right\|^2 \\ &= \mathrm{E}\left\|\sum_{j\in\mathbb{N}^d}\left(\exp(-\mu_j T)\cdot\langle\xi,h_j\rangle+\sum_{i\in\mathbb{N}^d}\lambda_i^{1/2}\cdot Z_{ij}(T)\right)\cdot h_j-\widehat{X}(T)\right\|^2. \quad (3.148)\end{aligned}$$

Given a vector $i \in \mathbb{N}^d$ of integers, a fixed time discretization $(t_{k,i})_{k\le n_i}$ of $[0,T]$ with $n_i \in \mathbb{N}$ and the evaluations of the scalar Brownian motions $\beta_i(t_{1,i}),\dots,\beta_i(t_{n_i,i})$, then we know that the conditional expectation

$$\widehat{Z}_{ij}(T) = \mathrm{E}(Z_{ij}(T)|\beta_i(t_{1,i}),\dots,\beta_i(t_{n_i,i}))$$

is the best approximation of $Z_{ij}(T)$ with $i,j\in\mathbb{N}^d$. Thus, with arbitrarily chosen non-empty, finite sets $\mathcal{I}$, $\mathcal{J}\subset\mathbb{N}^d$, sequences $(n_i)_{i\in\mathcal{I}}\in\mathbb{N}^{\mathcal{I}}$ and time discretizations $(t_{k,i})_{k\le n_i, i\in\mathcal{I}}$ of $[0,T]$, the best choice of $\widehat{X}(T)$ is of the form

$$\widehat{X}^*(T)=\sum_{j\in\mathcal{J}}\left(\exp(-\mu_j T)\cdot\langle\xi,h_j\rangle+\sum_{i\in\mathcal{I}}\lambda_i^{1/2}\cdot\widehat{Z}_{ij}(T)\right)\cdot h_j.$$

Hence, we obtain by $\widehat{X}(T)=\widehat{X}^*(T)$ in (3.148)

$$\begin{aligned} &\mathrm{E}\left\|X(T)-\widehat{X}(T)\right\|^2 \\ &\ge \mathrm{E}\left\|\sum_{j\notin\mathcal{J}}\left(\exp(-\mu_j T)\cdot\langle\xi,h_j\rangle+\sum_{i\in\mathbb{N}^d}\lambda_i^{1/2}\cdot Z_{ij}(T)\right)\cdot h_j\right. \\ &\qquad\left.+\sum_{j\in\mathcal{J}}\left(\sum_{i\in\mathcal{I}}\lambda_i^{1/2}\cdot(Z_{ij}(T)-\widehat{Z}_{ij}(T))+\sum_{i\notin\mathcal{I}}\lambda_i^{1/2}\cdot Z_{ij}(T)\right)\cdot h_j\right\|^2 \\ &= \mathrm{E}\sum_{k\in\mathbb{N}^d}\left(\sum_{j\notin\mathcal{J}}\left(\exp(-\mu_j T)\cdot\langle\xi,h_j\rangle+\sum_{i\in\mathbb{N}^d}\lambda_i^{1/2}\cdot Z_{ij}(T)\right)\cdot\langle h_j,h_k\rangle\right. \\ &\qquad\left.+\sum_{j\in\mathcal{J}}\left(\sum_{i\in\mathcal{I}}\lambda_i^{1/2}\cdot(Z_{ij}(T)-\widehat{Z}_{ij}(T))+\sum_{i\notin\mathcal{I}}\lambda_i^{1/2}\cdot Z_{ij}(T)\right)\cdot\langle h_j,h_k\rangle\right)^2\end{aligned}$$

using the Parseval equality and the continuity of the scalar product. Due to $\langle h_i, h_k\rangle \cdot \langle h_j, h_k\rangle = 0$ for every $k \in \mathbb{N}^d$ if $i \neq j$, $\mathrm{E}(Z_{ij}(T)) = 0$ for every $i, j \in \mathbb{N}^d$ and $(\beta_i)_{i\in\mathbb{N}^d}$ is an independent family of scalar Brownian motions, we conclude

$$\begin{aligned}
&\mathrm{E}\left\|X(T)-\widehat{X}(T)\right\|^2 \\
&\geq \sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}}\lambda_i\cdot\mathrm{E}\left(Z_{ij}(T)-\widehat{Z}_{ij}(T)\right)^2 \\
&\quad+\sum_{j\in\mathcal{J}}\mathrm{E}\left(\sum_{i\in\mathcal{I}}\lambda_i^{1/2}\cdot(Z_{ij}(T)-\widehat{Z}_{ij}(T))\sum_{\substack{k\in\mathcal{I}\\k\neq i}}\lambda_k^{1/2}\cdot(Z_{kj}(T)-\widehat{Z}_{kj}(T))\right) \\
&\quad+\sum_{j\in\mathcal{J}}\sum_{i\notin\mathcal{I}}\lambda_i\cdot\mathrm{E}Z_{ij}^2(T)+\sum_{j\notin\mathcal{J}}\sum_{i\in\mathbb{N}^d}\lambda_i\cdot\mathrm{E}Z_{ij}^2(T).
\end{aligned}$$

By (3.40) with (3.145) and $\widehat{\beta}_i(t) = \mathrm{E}(\beta_i(t)|\beta_i(t_{1,i}), \ldots, \beta_i(t_{n_i,i}))$, we have

$$\begin{aligned}
Z_{ij}(T)-\widehat{Z}_{ij}(T) = B_{ij}\cdot\Bigg(&\beta_i(T)-\mu_j\int_0^T\exp(-\mu_j(T-t))\cdot\beta_i(t)\,dt \\
&-\widehat{\beta}_i(T)+\mu_j\int_0^T\exp(-\mu_j(T-t))\cdot\widehat{\beta}_i(t)\,dt\Bigg).
\end{aligned} \tag{3.149}$$

Note, that the conditional expectation of a scalar Brownian motion $\beta_i = (\beta_i(t))_{t\geq 0}$, given its evaluations at the time nodes $(t_{k,i})_{k\leq n_i}$, is derived by piecewise linear interpolation, i.e. for $t \in [t_{k,i}, t_{k+1,i}]$,

$$\widehat{\beta}_i(t) = \beta_i(t_{k,i}) + \frac{t-t_{k,i}}{t_{k+1,i}-t_{k,i}}(\beta_i(t_{k+1,i})-\beta_i(t_{k,i})).$$

Because $(\beta_i)_{i\in\mathbb{N}^d}$ is an independent family of scalar Brownian motions and due to (3.149), the error estimation reduces to

$$\begin{aligned}
\mathrm{E}\left\|X(T)-\widehat{X}(T)\right\|^2 &\geq \sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}}\lambda_i\cdot\mathrm{E}\left(Z_{ij}(T)-\widehat{Z}_{ij}(T)\right)^2 \\
&\quad+\sum_{j\in\mathcal{J}}\sum_{i\notin\mathcal{I}}\lambda_i\cdot\mathrm{E}Z_{ij}^2(T)+\sum_{j\notin\mathcal{J}}\sum_{i\in\mathbb{N}^d}\lambda_i\cdot\mathrm{E}Z_{ij}^2(T) \\
&\geq \sum_{i\in\mathcal{J}\cap\mathcal{I}}\lambda_i\cdot\mathrm{E}\left(Z_{ii}(T)-\widehat{Z}_{ii}(T)\right)^2 \\
&\quad+\sum_{i\notin\mathcal{J}\cap\mathcal{I}}\lambda_i\cdot\mathrm{E}Z_{ii}^2(T). \qquad (3.150)
\end{aligned}$$

Remember from (3.15) and (3.39), that $Z_{ii}(T)=B_{ii}\cdot Y_i(T)$ and $\widehat{Z}_{ii}(T)=B_{ii}\cdot\widehat{Y}_i(T)$. Thus, from (3.3), we have

$$\mathrm{E}\left\|X(T)-\widehat{X}(T)\right\|^2\succeq\sum_{i\in\mathcal{J}\cap\mathcal{I}}\lambda_i\cdot\mathrm{E}\left(Y_i(T)-\widehat{Y}_i(T)\right)^2+\sum_{i\notin\mathcal{J}\cap\mathcal{I}}\lambda_i\cdot\mathrm{E}Y_i^2(T)$$

and the proposition follows by the proof of Proposition 3.4.2, starting with the estimation (3.106). □

Comparing the Propositions 3.4.3, 3.4.4, 3.4.5 and 3.4.6 with Proposition 3.4.7, we see, that for some choices of the parameters α, β, γ and d, we obtain asymptotic optimality for the constructed algorithms $\widehat{X}_N^{\mathrm{uni}}(T)\in\mathfrak{X}_{c\cdot N}^{\mathrm{uni}}$ and $\widehat{X}_N^{\#}(T)\in\mathfrak{X}_{c\cdot N}^{\#}$. Thus, we obtain the both theorems in Section 3.3.

Proof of Theorem 3.3.1

The theorem is proved by combining the Propositions 3.4.3, 3.4.4, 3.4.5 and 3.4.7. □

Proof of Theorem 3.3.2

The theorem is proved by combining the Propositions 3.4.3, 3.4.4, 3.4.6 and 3.4.7. □

Chapter 4

Numerical Results

In this chapter we visualize and compare simulated trajectories of algorithms constructed in the Sections 3.2 and 3.3 that approximately solve (3.1). Moreover, we approximately compute the error of two concrete approximation schemes introduced in Section 3.3 using Monte Carlo experiments and compare them to the theoretical estimates. For this purpose in the whole chapter we consider the basis functions

$$h_j(u) = 2^{d/2} \cdot \prod_{\ell=1}^{d} \sin(j_\ell \cdot \pi \cdot u_\ell), \quad u \in (0,1)^d,$$

of H as the normalized eigenfunctions of A and Q with the corresponding eigenvalues

$$\mu_j = \pi^2 \cdot |j|_2^2$$

of A and

$$\lambda_j = |j|_2^{-\gamma}$$

of Q for $d \in \mathbb{N}$, $j \in \mathbb{N}^d$ and the real-valued parameter γ. Note that the considered eigenfunctions and eigenvalues of A coincide with those of the Laplace operator Δ on the unit cube with Dirichlet boundary conditions. That means we have $\alpha = 2$ in our Assumption 3.0.3 on A. Throughout this chapter we also set $\xi = 0$ and $T = 1$. Furthermore, we assume that the diffusion B is a time-independent pointwise multiplication operator, i.e.

$$B(t)h = g \cdot h$$

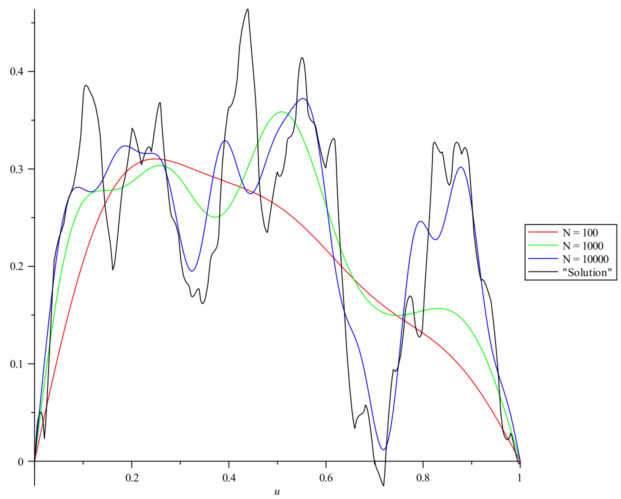

Figure 4.1: Realizations of $\widehat{X}_N^{\mathrm{uni}}(1)$ for $d = 1$, $\gamma = 0$ and $g(u) = 1$

for $t \geq 0$ and $h \in H$ with $g \in C^1([0,1]^d)$. Now, we compute realizations $\widehat{x}_N^{\diamond}(1)$ of $\widehat{X}_N^{\diamond}(1)$ with $\diamond \in \{\mathrm{uni}, \mathrm{equi}, \#, *\}$ for different values of N. All those realizations use the same trajectory of the driving (cylindrical) Wiener process W for comparison.

In Figures 4.1 to 4.4 we consider $d = 1$, $\gamma = 0$ and $g(u) = 1$ for the computation. That means we choose parameters used for the stochastic heat equation with the identity operator as diffusion in the space-time white noise case. Therefore, we here compare trajectories of the algorithms constructed in Section 3.2 in the (ID) case. In every one of those figures, we plot and compare computed realizations for one the respective approximations (3.21), (3.22), (3.26) and (3.29) using $N = 100$, $N = 1000$ as well as $N = 10000$ evaluations of the scalar Brownian motions. Furthermore, we always plot the corresponding realization $\widehat{x}_N^*(1)$ of $\widehat{X}_N^*(1)$ with $N = 10000$ as a substitute for the exact solution of the equation. We see that the algorithms using non-equidistant time discretizations give a far better approximation than the algorithms based on equidistant time nodes. The realizations $\widehat{x}_{10000}^{\mathrm{equi}}(1)$ and $\widehat{x}_{10000}^{\mathrm{uni}}(1)$ only have roughly the same behaviour of $\widehat{x}_{10000}^*(1)$ while $\widehat{x}_{10000}^{\#}(1)$ and even $\widehat{x}_{1000}^*(1)$ actually provide much of its local details.

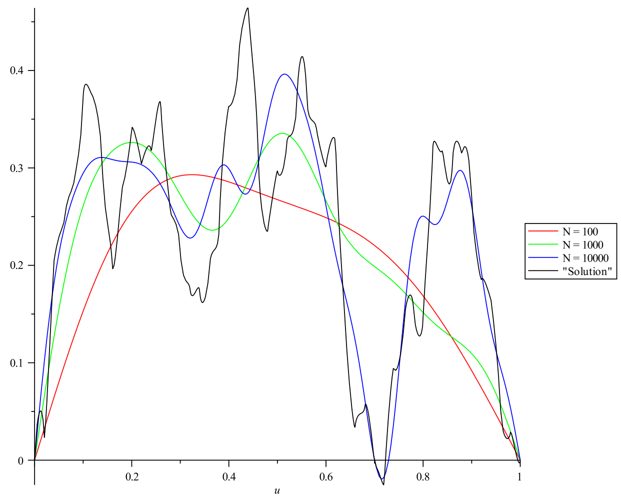

Figure 4.2: Realizations of $\widehat{X}_N^{\mathrm{equi}}(1)$ for $d = 1$, $\gamma = 0$ and $g(u) = 1$

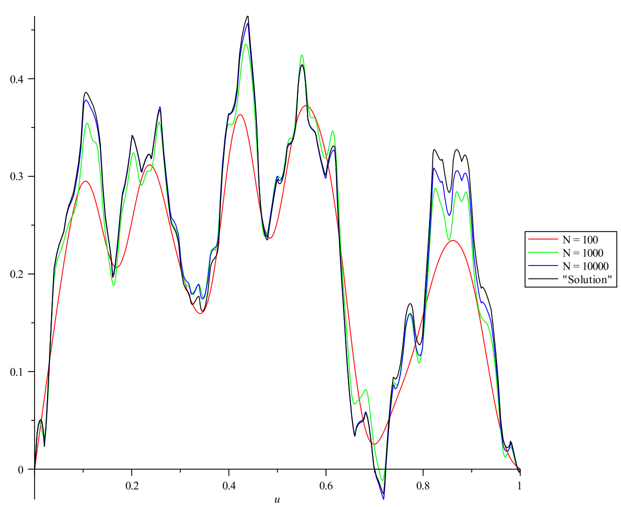

Figure 4.3: Realizations of $\widehat{X}_N^{\#}(1)$ for $d = 1$, $\gamma = 0$ and $g(u) = 1$

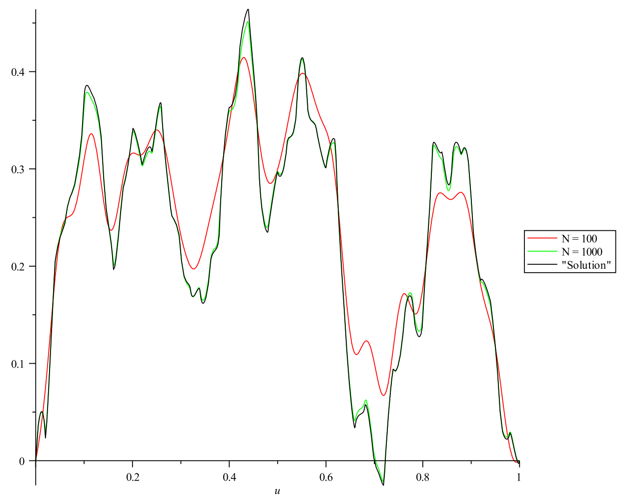

Figure 4.4: Realizations of $\widehat{X}^*_N(1)$ for $d = 1$, $\gamma = 0$ and $g(u) = 1$

For the computations of the Figures 4.5 to 4.8 we consider $d = 2$, $\gamma = 2.1$ and $g(u_1, u_2) = 1$. Therefore, we have parameters for a stochastic heat equation with the identity operator as diffusion in the nuclear noise case with a smaller smoothness. We respectively plot one realization $\widehat{x}^\diamond_N(1)$ with $N = 10000$ for every $\diamond \in \{\mathrm{uni}, \mathrm{equi}, \#, *\}$ in one figure. As in the (ID) case, we see that $\widehat{x}^\#_{10000}(1)$ and $\widehat{x}^*_{10000}(1)$ provide much more local details than $\widehat{x}^{\mathrm{uni}}_{10000}(1)$ and $\widehat{x}^{\mathrm{equi}}_{10000}(1)$. That again indicates that $\widehat{X}^*_N(1)$ and $\widehat{X}^\#_N(1)$ are the better approximations.

In Figures 4.9 to 4.12, we turn to equations with $d = 1$ and $g(u) = \exp(u)$. Thus, we set $\beta = 2$ in the Assumption 3.0.2 on B. We compute realizations $\widehat{x}^\diamond_N(1)$ of the corresponding algorithms $\widehat{X}^\diamond_N(1)$ with $\diamond \in \{\mathrm{uni}, \#\}$ established in Section 3.3. Here we show trajectories of the same approximation scheme using respectively $N = 1000$, $N = 10000$ and $N = 100000$ evaluations of the scalar Brownian motions in every plot. We also insert the corresponding realization $\widehat{x}^\#_{100000}(1)$ of (3.75) as a substitute for the realization of the exact solution. In Figures 4.9 and 4.10, we study the (ID) case and see that $\widehat{x}^\#_{100000}(1)$ provides more local details than $\widehat{x}^{\mathrm{uni}}_{100000}(1)$. The (TC) case with $\gamma = 1.1$ is shown in Figures 4.11 and 4.12. Here we observe that $\widehat{x}^{\mathrm{uni}}_{100000}(1)$ only shows an irregular behaviour of $\widehat{x}^\#_{100000}(1)$, while $\widehat{x}^\#_{10000}(1)$ already gives more local details.

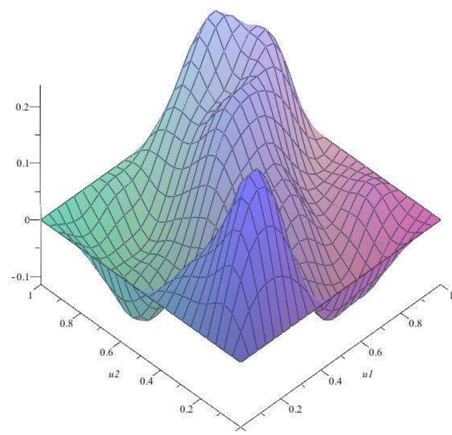

Figure 4.5: Realization of $\widehat{X}^{\mathrm{uni}}_{10000}(1)$ for $d = 2$, $\gamma = 2.1$ and $g(u_1, u_2) = 1$

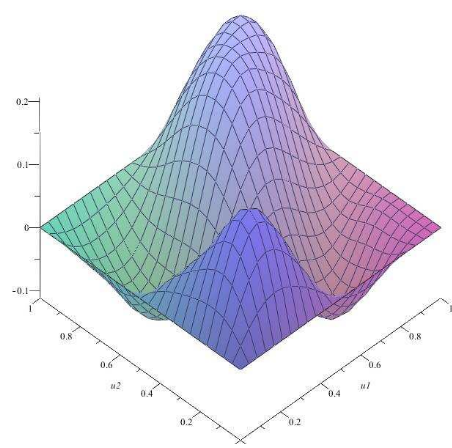

Figure 4.6: Realization of $\widehat{X}^{\mathrm{equi}}_{10000}(1)$ for $d = 2$, $\gamma = 2.1$ and $g(u_1, u_2) = 1$

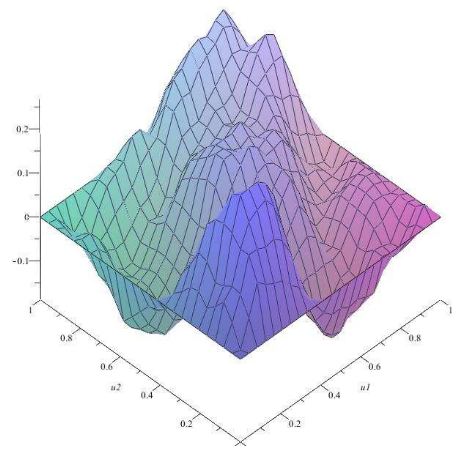

Figure 4.7: Realization of $\widehat{X}^{\#}_{10000}(1)$ for $d = 2$, $\gamma = 2.1$ and $g(u_1, u_2) = 1$

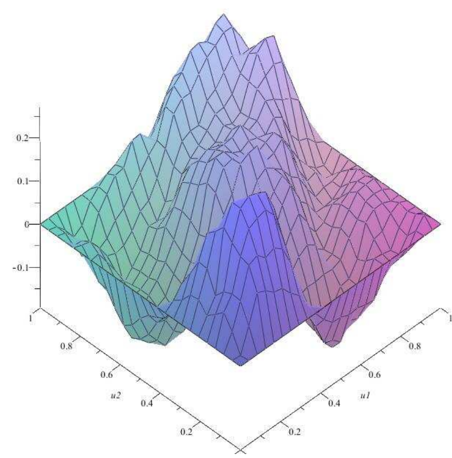

Figure 4.8: Realization of $\widehat{X}^{*}_{10000}(1)$ for $d = 2$, $\gamma = 2.1$ and $g(u_1, u_2) = 1$

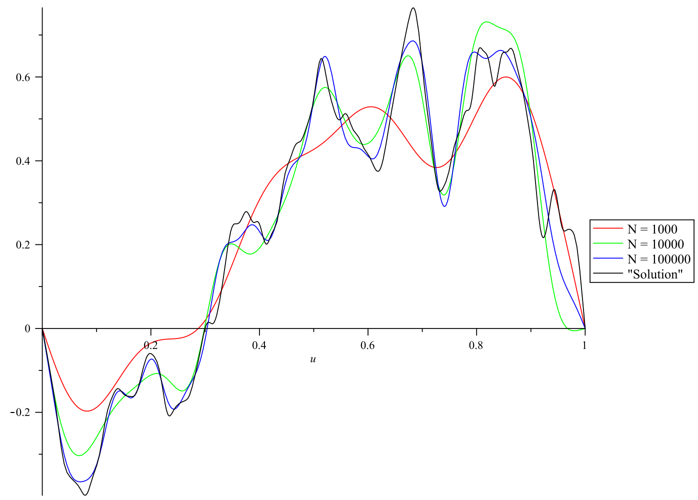

Figure 4.9: Realizations of $\widehat{X}_N^{\mathrm{uni}}(1)$ for $d = 1$, $\gamma = 0$ and $g(u) = \exp(u)$

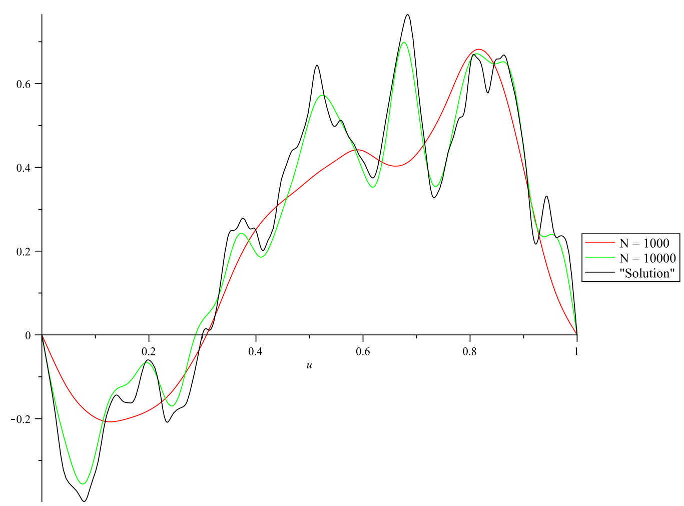

Figure 4.10: Realizations of $\widehat{X}_N^{\#}(1)$ for $d = 1$, $\gamma = 0$ and $g(u) = \exp(u)$

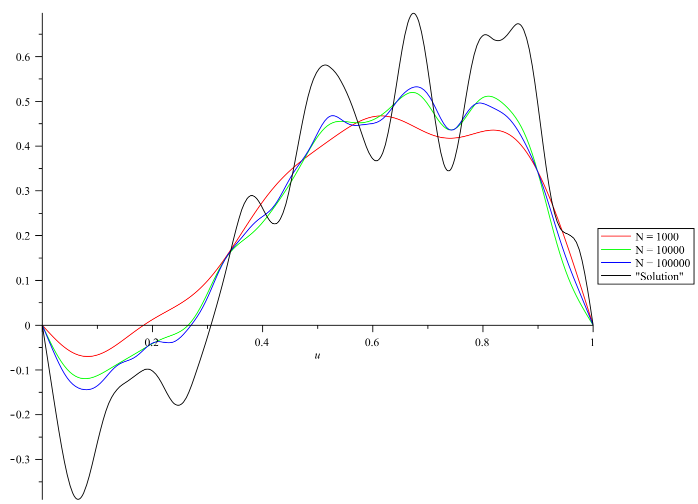

Figure 4.11: Realizations of $\widehat{X}_N^{\mathrm{uni}}(1)$ for $d = 1$, $\gamma = 1.1$ and $g(u) = \exp(u)$

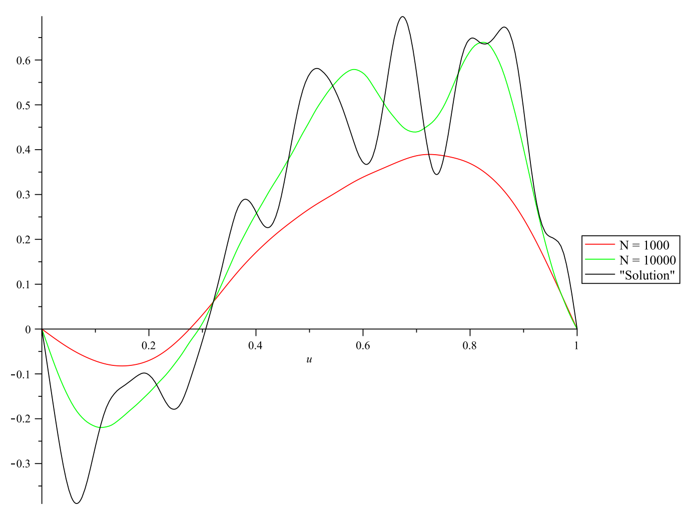

Figure 4.12: Realizations of $\widehat{X}_N^{\#}(1)$ for $d = 1$, $\gamma = 1.1$ and $g(u) = \exp(u)$

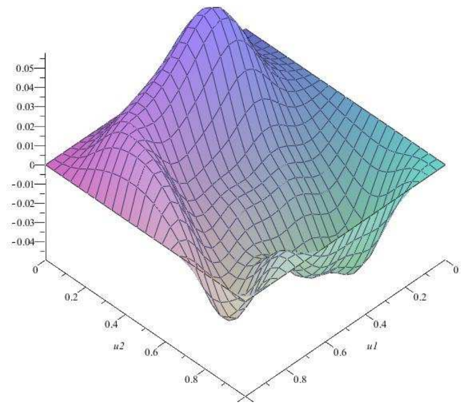

Figure 4.13: Realization of $\widehat{X}^{\#}_{100000}(1)$ for $d = 2$, $\gamma = 2.1$ and $g(u_1, u_2) = u_1 + u_2$

The Figures 4.13 to 4.15 attend to the setting $d = 2$ and $g(u_1, u_2) = u_1 + u_2$. We show realizations $\widehat{x}^{\#}_{100000}(1)$ of $\widehat{X}^{\#}_{100000}(1)$ for $\gamma = 2.1$ and $\gamma = 4.1$ in the first two figures and clearly see more local details in the first plot. The third Figure 4.15 gives $\widehat{x}^{\mathrm{uni}}_{100000}(1)$ with $\gamma = 4.1$ and there is no notable difference to Figure 4.14. This suggests that the approximations $\widehat{X}^{\#}_{N}(1)$ and $\widehat{X}^{\mathrm{uni}}_{N}(1)$ are of the same quality in the latter setting, which is not contradictory to our results.

Furthermore in this chapter, we use a Monte Carlo simulation to compute the errors $\mathrm{e}(\widehat{X}^{\mathrm{uni}}_{N}(1))$ and $\mathrm{e}(\widehat{X}^{\#}_{N}(1))$ of the respective approximation schemes (3.52) and (3.75) established in Section 3.3 for coupled systems of equations in the space-time white noise case. Here we study equations with either $g(u) = u$ or $g(u) = \exp(u)$ and we use the approach introduced in Section 9.3 in [KP92]. For an arbitrary approximation $\widehat{X}_N(1)$ of the mild solution $X(1)$, we compute the error

$$\mathrm{e}\left(\widehat{X}_N(1)\right) = \left(\mathrm{E}\left\|X(1) - \widehat{X}_N(1)\right\|^2\right)^{1/2} \tag{4.1}$$

by Monte Carlo experiments in the following way. Usually, we cannot calculate the

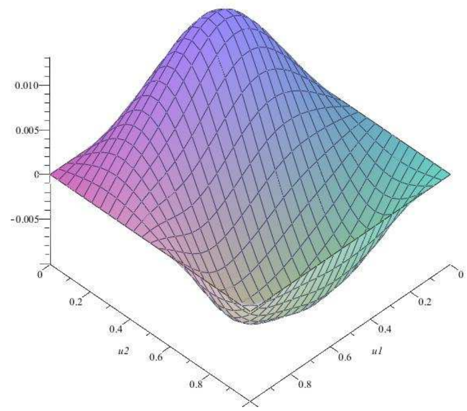

Figure 4.14: Realization of $\widehat{X}^{\#}_{100000}(1)$ for $d = 2$, $\gamma = 4.1$ and $g(u_1, u_2) = u_1 + u_2$

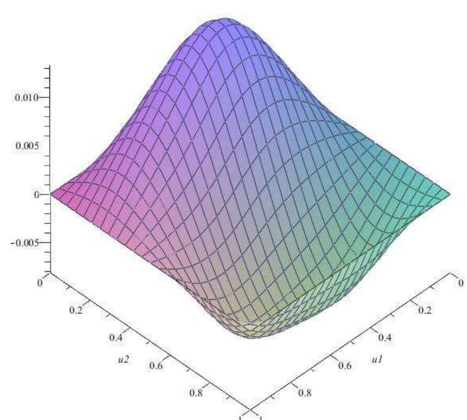

Figure 4.15: Realization of $\widehat{X}^{\mathrm{uni}}_{100000}(1)$ for $d = 2$, $\gamma = 4.1$ and $g(u_1, u_2) = u_1 + u_2$

mild solution explicitly. Therefore, we use here the algorithm (3.75) with $\widetilde{N} = 100000$ evaluations of the scalar Brownian motions as a substitute $\widetilde{X}(1)$ for $X(1)$ in (4.1) and $\widetilde{N} \gg N$. Now, we can repeat L independent simulations of realizations of $\widetilde{X}(1)$ and $\widehat{X}_N(1)$ corresponding to the same trajectories of the driving (cylindrical) Wiener process. We denote the respective kth computed realization by $\widetilde{X}_k(1)$ and $\widehat{X}_{N,k}(1)$. Thus,

$$\widehat{\mathrm{e}}_{L,N} = \left(\frac{1}{L} \sum_{k=1}^{L} \left\| \widetilde{X}_k(1) - \widehat{X}_{N,k}(1) \right\|^2 \right)^{1/2}$$

is an estimation for (4.1). In addition, we estimate the variance $\widehat{\sigma}^2$ of $\widehat{\mathrm{e}}_{L,N}$ and use it to construct a confidence interval for the error $\mathrm{e}(\widehat{X}_N(1))$. For this reason, we group the simulations into M batches of L simulations each and estimate the variance in the following way. Let $\widetilde{X}_{k,j}(1)$ be the value in H of the kth generated trajectory of the solution substitute in the jth batch and $\widehat{X}_{N,k,j}(1)$ be its approximation. Now, let

$$\widehat{\mathrm{e}}_{L,M,N,j} = \left(\frac{1}{L} \sum_{k=1}^{L} \left\| \widetilde{X}_{k,j}(1) - \widehat{X}_{N,k,j}(1) \right\|^2 \right)^{1/2}$$

be the independent average errors of the M batches $j = 1, \ldots, M$. The mean of the batch averages is estimated by

$$\widehat{\mathrm{e}}_{L,M,N} = \frac{1}{M} \sum_{j=1}^{M} \widehat{\mathrm{e}}_{L,M,N,j}$$

and we use

$$\widehat{\sigma}^2_{L,M,N} = \frac{1}{M-1} \sum_{j=1}^{M} \left(\widehat{\mathrm{e}}_{L,M,N,j} - \widehat{\mathrm{e}}_{L,M,N}\right)^2$$

to estimate the variance $\widehat{\sigma}^2$ of the batch averages. For batch sizes $L \geq 15$ the batch averages can be interpreted as being Gaussian. Thus, we use the Student t-distribution to compute confidence intervals for a sum of independent approximately Gaussian distributed random variables with unknown variance. For the Student t-distribution with $M-1$ degrees of freedom the $100(1-\alpha)\%$ confidence interval for $\mathrm{e}(\widehat{X}_N(1))$ has the form

$$\left(\widehat{\mathrm{e}}_{L,M,N} - \Delta\widehat{\mathrm{e}}_{L,M,N}, \widehat{\mathrm{e}}_{L,M,N} + \Delta\widehat{\mathrm{e}}_{L,M,N}\right)$$

with

$$\Delta\widehat{\mathrm{e}}_{L,M,N} = t_{M-1,1-\alpha/2} \cdot \left(\widehat{\sigma}^2_{L,M,N}/M\right)^{1/2}$$

where $t_{M-1,1-\alpha/2}$ is determined from the Student t-distribution with $M-1$ degrees of freedom. In the Figures 4.16 and 4.17, we show computed values of $\log_{10}(\widehat{e}_{L,M,N})$ with the corresponding confidence intervals as a function of $\log_{10}(N)$ for the algorithms $\widehat{X}_N^{\mathrm{uni}}(1)$ and $\widehat{X}_N^{\#}(1)$ in place of $\widehat{X}_N(1)$. We always choose $L = 50$, $M = 20$, $\alpha = 0.05$ and the error estimates are calculated for $N = 100, 500, 1000, 5000, 10000$. Furthermore, we include the linear regression line with respect to the logarithmic error estimates.

In Figure 4.16, we consider $g(u) = u$. Here the slopes of the regression lines of the error estimates for $\widehat{X}_N^{\mathrm{uni}}(1)$ and $\widehat{X}_N^{\#}(1)$ are about -0.1997 and -0.3416. The confidence intervals for $\mathrm{e}(\widehat{X}_N^{\mathrm{uni}}(1))$ are slightly larger than the ones for $\mathrm{e}(\widehat{X}_N^{\#}(1))$. It seems that for higher cost the error of the approximation using an uniform time discretization is larger than the error of the approximation with non-equidistant time nodes. Hence, it appears that the approximation $\widehat{X}_N^{\#}(1)$ is superior to the approximation $\widehat{X}_N^{\mathrm{uni}}(1)$, which coincides with our theoretical result. The same conclusion follows by Figure 4.17 in the case $g(u) = \exp(u)$. Here the slopes of the respective regression lines are about -0.2163 for the approximation $\widehat{X}_N^{\mathrm{uni}}(1)$ and -0.3358 for the approximation $\widehat{X}_N^{\#}(1)$ with slightly smaller confidence intervals for $\mathrm{e}(\widehat{X}_N^{\#}(1))$ as for $\mathrm{e}(\widehat{X}_N^{\mathrm{uni}}(1))$. For completion, the Tables 4.1 to 4.4 show the computed values used in the Figures 4.16 and 4.17.

We refer to Section 5.4 in [W08] for a more detailed statistical analysis of the numerically computed errors of some approximation schemes for the stochastic heat equation with the identity operator as diffusion in the (ID) case. See Section 5.1 in [MGRW08] for error estimation using explicit error formulas instead of Monte Carlo simulations for that stochastic heat equation in both cases (TC) and (ID). By this approach, the average error of specific algorithms can numerically computed up to any accuracy.

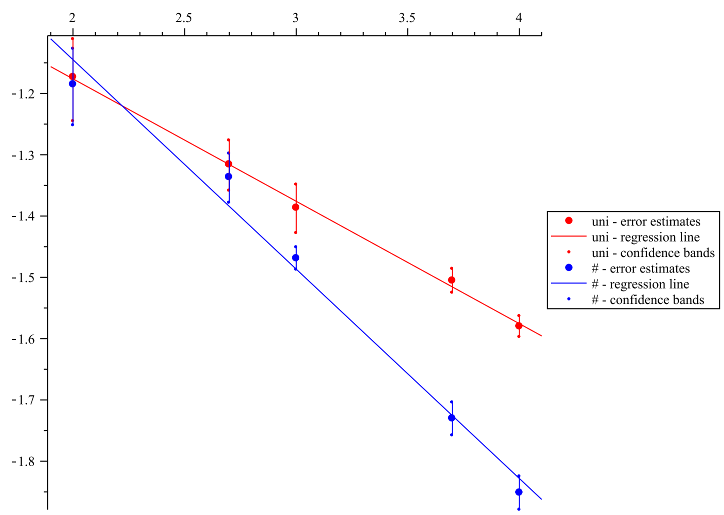

Figure 4.16: Error computation of $\mathrm{e}(\widehat{X}_N^{\#}(1))$ and $\mathrm{e}(\widehat{X}_N^{\mathrm{uni}}(1))$ for $g(u) = u$

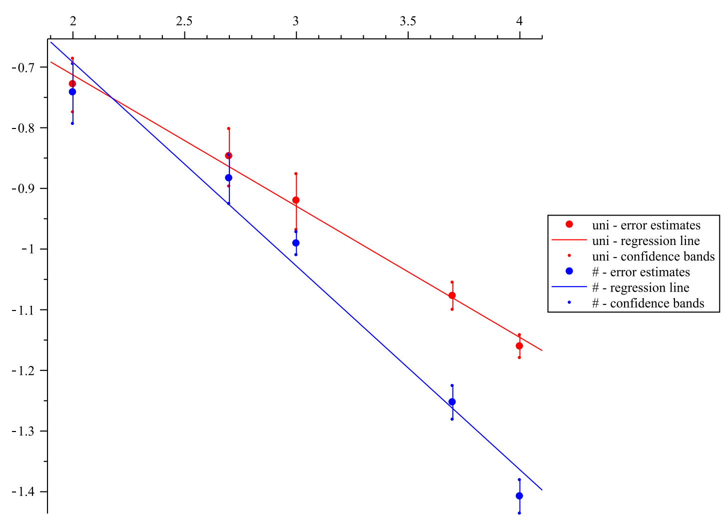

Figure 4.17: Error computation of $\mathrm{e}(\widehat{X}_N^{\#}(1))$ and $\mathrm{e}(\widehat{X}_N^{\mathrm{uni}}(1))$ for $g(u) = \exp(u)$

N	$\widehat{\mathrm{e}}_{L,M,N}$	$\widehat{\sigma}^2_{L,M,N}$	$\Delta\widehat{\mathrm{e}}_{L,M,N}$	$\widehat{\mathrm{e}}_{L,M,N} - \Delta\widehat{\mathrm{e}}_{L,M,N}$	$\widehat{\mathrm{e}}_{L,M,N} + \Delta\widehat{\mathrm{e}}_{L,M,N}$
100	0.06713	0.0004815845	0.01026	0.05687	0.07739
500	0.04835	0.0000949027	0.00455	0.04380	0.05290
1000	0.04108	0.0000637856	0.00373	0.03735	0.04481
5000	0.03125	0.0000089530	0.00140	0.02985	0.03265
10000	0.02631	0.0000048572	0.00103	0.02528	0.02734

Table 4.1: Computed values for $\mathrm{e}(\widehat{X}^{\mathrm{uni}}_N(1))$ with $g(u) = u$

N	$\widehat{\mathrm{e}}_{L,M,N}$	$\widehat{\sigma}^2_{L,M,N}$	$\Delta\widehat{\mathrm{e}}_{L,M,N}$	$\widehat{\mathrm{e}}_{L,M,N} - \Delta\widehat{\mathrm{e}}_{L,M,N}$	$\widehat{\mathrm{e}}_{L,M,N} + \Delta\widehat{\mathrm{e}}_{L,M,N}$
100	0.06532	0.003979405	0.00932	0.05600	0.07464
500	0.04610	0.0000827286	0.00425	0.04185	0.05035
1000	0.03398	0.0000093742	0.00143	0.03255	0.03541
5000	0.01862	0.0000060169	0.00115	0.01747	0.01977
10000	0.01409	0.0000035824	0.00088	0.01321	0.01497

Table 4.2: Computed values for $\mathrm{e}(\widehat{X}^{\#}_N(1))$ with $g(u) = u$

N	$\widehat{\mathrm{e}}_{L,M,N}$	$\widehat{\sigma}^2_{L,M,N}$	$\Delta\widehat{\mathrm{e}}_{L,M,N}$	$\widehat{\mathrm{e}}_{L,M,N} - \Delta\widehat{\mathrm{e}}_{L,M,N}$	$\widehat{\mathrm{e}}_{L,M,N} + \Delta\widehat{\mathrm{e}}_{L,M,N}$
100	0.18698	0.0016419562	0.01894	0.16804	0.20592
500	0.14227	0.0010949261	0.01546	0.12681	0.15773
1000	0.12019	0.0007368568	0.01269	0.10750	0.13288
5000	0.08371	0.0000851532	0.00431	0.07940	0.08802
10000	0.06912	0.0000410136	0.00299	0.06613	0.07211

Table 4.3: Computed values for $\mathrm{e}(\widehat{X}^{\mathrm{uni}}_N(1))$ with $g(u) = \exp(u)$

N	$\widehat{\mathrm{e}}_{L,M,N}$	$\widehat{\sigma}^2_{L,M,N}$	$\Delta\widehat{\mathrm{e}}_{L,M,N}$	$\widehat{\mathrm{e}}_{L,M,N} - \Delta\widehat{\mathrm{e}}_{L,M,N}$	$\widehat{\mathrm{e}}_{L,M,N} + \Delta\widehat{\mathrm{e}}_{L,M,N}$
100	0.18131	0.0019282031	0.02052	0.16079	0.20183
500	0.13084	0.0006788790	0.01218	0.11866	0.14302
1000	0.10216	0.0000905409	0.00445	0.09771	0.10661
5000	0.05590	0.0000581658	0.00357	0.05233	0.05947
10000	0.03912	0.0000281340	0.00248	0.03664	0.04160

Table 4.4: Computed values for $\mathrm{e}(\widehat{X}_N^{\#}(1))$ with $g(u) = \exp(u)$

Appendix A

Bounded Linear Operators

In this appendix we recall some definitions and basic properties for bounded linear operators that are used throughout the thesis. For more details see, e.g., [W07], the following descriptions are mainly taken from. Throughout this appendix, let $\mathfrak{I}$ be a countable index set and consider the two separable real Hilbert spaces $(G, \|\cdot\|_G, \langle\cdot,\cdot\rangle_G)$ and $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$. We denote the class of all bounded linear operators from G to H by $\mathcal{L}(G,H)$ and the class of all compact operators from G to H by $\mathcal{L}_C(G,H)$. For simplicity we define $\mathcal{L}(H,H) = \mathcal{L}(H)$ and $\mathcal{L}_C(H,H) = \mathcal{L}_C(H)$. Note that we call $A^* \in \mathcal{L}(H,G)$ the adjoint operator of $A \in \mathcal{L}(G,H)$, which means $\langle A^*h, g\rangle_G = \langle h, Ag\rangle_H$ for every $g \in G$ and $h \in H$. If $G = H$ and $\langle Ah_1, h_2\rangle_H = \langle h_1, Ah_2\rangle_H$ for every $h_1, h_2 \in H$, we call $A \in \mathcal{L}(H)$ symmetric. Moreover, $A \in \mathcal{L}(H)$ is called non-negative, if $\langle Ah, h\rangle \geq 0$ for every $h \in H$. $\|A\|_{\mathcal{L}(G,H)} = \sup_{\|g\|_G \leq 1} \|Ag\|_H$ defines a norm on $\mathcal{L}(G,H)$, which is called the operator norm.

Definition A.0.1 ***(Hilbert-Schmidt operator)***
An operator $A \in \mathcal{L}(G,H)$ is called a Hilbert-Schmidt operator from G to H, if there exists an orthonormal basis $(g_i)_{i\in\mathfrak{I}}$ of G such that

$$\left(\sum_{i\in\mathfrak{I}} \|Ag_i\|_H^2\right)^{1/2} < \infty.$$

We denote the class of all Hilbert-Schmidt operators from G to H by $\mathcal{L}_{\mathrm{HS}}(G,H)$ and in the case of $G = H$ by $\mathcal{L}_{\mathrm{HS}}(H)$. Furthermore, for $A \in \mathcal{L}_{\mathrm{HS}}(G,H)$ we define

$$\|A\|_{\mathrm{HS}} = \left(\sum_{i \in \mathfrak{I}} \|Ag_i\|_H^2 \right)^{1/2}.$$

The number $\|A\|_{\mathrm{HS}}$ does not depend on the choice of the orthonormal basis $(g_i)_{i\in\mathfrak{I}}$ of G and $\|\cdot\|_{\mathrm{HS}}$ defines a norm on $\mathcal{L}_{\mathrm{HS}}(G,H)$, which is called the Hilbert-Schmidt norm.

Proposition A.0.8 *Let $A, B \in \mathcal{L}_{\mathrm{HS}}(G,H)$. Then the following properties hold.*

i) $(\mathcal{L}_{\mathrm{HS}}(G,H), \|\cdot\|_{\mathrm{HS}}, \langle\cdot,\cdot\rangle_{\mathrm{HS}})$ *is a separable Hilbert space with the scalar product*

$$\langle A, B\rangle_{\mathrm{HS}} = \sum_{i\in\mathfrak{I}} \langle Ag_i, Bg_i\rangle_H .$$

ii) $C \in \mathcal{L}(G,H)$ *is a Hilbert-Schmidt operator if and only if* $C^* \in \mathcal{L}(H,G)$ *is a Hilbert-Schmidt operator. In this case, it holds* $\|C\|_{\mathcal{L}(G,H)} \le \|C\|_{\mathrm{HS}} = \|C^*\|_{\mathrm{HS}}$.

iii) Let K be another separable real Hilbert space and suppose that $C_1 \in \mathcal{L}_{\mathrm{HS}}(G,H)$, $C_2 \in \mathcal{L}(H,K)$ or $C_1 \in \mathcal{L}(G,H)$, $C_2 \in \mathcal{L}_{\mathrm{HS}}(H,K)$. Then $C_2C_1 \in \mathcal{L}_{\mathrm{HS}}(G,K)$ and $\|C_2C_1\| \le \|C_2\| \cdot \|C_1\|$ with respect to the corresponding norms.

Proof: See, e.g., Section 1.2. in [KX95], Appendix B in [PR07] and Section VI.6 in [W07]. □

Definition A.0.2 ***(Nuclear operator)***
The operator $A \in \mathcal{L}(G,H)$ is called a nuclear operator from G to H, if there exists an orthonormal basis $(g_i)_{i\in\mathfrak{I}}$ of G such that

$$\sum_{i\in\mathfrak{I}} \|Ag_i\|_H < \infty.$$

We denote the class of all nuclear operators from G to H by $\mathcal{L}_{\mathrm{nuc}}(G,H)$ and in the case of $G = H$ by $\mathcal{L}_{\mathrm{nuc}}(H)$.

Proposition A.0.9 *Let $A \in \mathcal{L}_{\mathrm{nuc}}(G,H)$. Then the following properties hold.*

i)

$$\|A\|_{\text{nuc}} = \inf\left\{ \sum_{i\in\mathfrak{I}} \|Ag_i\|_H \;\middle|\; (g_i)_{i\in\mathcal{I}} \textit{ is an orthonormal basis of } G \right\}$$

defines a norm in $\mathcal{L}_{\text{nuc}}(G,H)$ *and* $(\mathcal{L}_{\text{nuc}}(G,H), \|\cdot\|_{\text{nuc}})$ *is a Banach space.*

ii) If $G = H$*, the trace of* A*,*

$$\operatorname{tr}(A) = \sum_{i\in\mathfrak{I}} \langle Ah_i, h_i\rangle_H\,,$$

does not depend on the choice of the orthonormal basis $(h_i)_{i\in\mathfrak{I}}$ *of* H *and* $|\operatorname{tr}(A)| \leq \|A\|_{\text{nuc}}$*. Moreover, if* $B \in \mathcal{L}(H)$*, then* $AB, BA \in \mathcal{L}_{\text{nuc}}(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA) \leq \|A\|_{\text{nuc}} \cdot \|B\|_{\mathcal{L}(H)}$*.*

iii) $B \in \mathcal{L}(H)$ *is a nuclear operator if and only if* $B^* \in \mathcal{L}(H)$ *is a nuclear operator. In this case, it holds* $\operatorname{tr}(B) = \operatorname{tr}(B^*)$*.*

iv) Let K *be another separable real Hilbert space and suppose that* $C_1 \in \mathcal{L}_{\text{nuc}}(G,H)$*,* $C_2 \in \mathcal{L}(H,K)$ *or* $C_1 \in \mathcal{L}(G,H)$*,* $C_2 \in \mathcal{L}_{\text{nuc}}(H,K)$*. Then* $C_2C_1 \in \mathcal{L}_{\text{nuc}}(G,K)$*.*

v) It holds $\mathcal{L}_{\text{nuc}}(G,H) \subset \mathcal{L}_{\text{HS}}(G,H) \subset \mathcal{L}_C(G,H) \subset \mathcal{L}(G,H)$ *with the estimation* $\|A\|_{\mathcal{L}(G,H)} \leq \|A\|_{\text{HS}} \leq \|A\|_{\text{nuc}}$*.*

vi) Let K *be another separable real Hilbert space and suppose that* $C_1 \in \mathcal{L}_{\text{HS}}(G,H)$ *and* $C_2 \in \mathcal{L}_{\text{HS}}(H,K)$*. Then* $C_2C_1 \in \mathcal{L}_{\text{nuc}}(G,K)$ *and* $\|C_2C_1\|_{\text{nuc}} \leq \|C_2\|_{\text{HS}} \cdot \|C_1\|_{\text{HS}}$*.*

Proof: See, e.g., Appendix C in [DPZ92], Section 1.2. in [KX95], Appendix B in [PR07] and Section VI.5 in [W07]. □

Definition A.0.3 ***(Trace class operator)***
A non-negative and symmetric operator $A \in \mathcal{L}_{\text{nuc}}(H)$ *is called trace class operator.*

Proposition A.0.10 *Let* $A \in \mathcal{L}(H)$ *be a non-negative and symmetric operator. Then the following properties hold.*

i) There exists exactly one non-negative and symmetric operator $A^{1/2} \in \mathcal{L}(H)$ *such that* $A^{1/2} \circ A^{1/2} = A$*. If, in addition,* $\operatorname{tr}(A) < \infty$*, then it holds* $A^{1/2} \in \mathcal{L}_{\text{HS}}(H)$ *with* $\|A^{1/2}\|_{\text{HS}}^2 = \operatorname{tr}(A)$*.*

ii) The operator A is a nuclear operator if and only if for an orthonormal basis $(h_i)_{i\in\mathfrak{I}}$ of H holds

$$\sum_{i\in\mathfrak{I}} \langle Ah_i, h_i\rangle < \infty.$$

In this case, it holds $\mathrm{tr}(A) = \|A\|_{\mathrm{nuc}}$ and there exists an orthonormal basis $(e_i)_{i\in\mathfrak{I}}$ of H such that

$$Ae_i = \lambda_i \cdot e_i$$

with $\lambda_i \geq 0$ for every $i \in \mathfrak{I}$ and 0 is the only accumulation point of the sequence $(\lambda_i)_{i\in\mathfrak{I}}$.

Proof: See, e.g., Appendix C in [DPZ92] and Sections 2.1. and 2.3. in [PR07]. $\square$

Appendix B

Semigroups of Linear Operators

In this appendix we shortly summarize the definitions and results for the semigroups and their generators used in Chapters 2 and 3. In addition, we present an important example of such a generator, which we also may consider as operator in the drift term by Assumption 3.0.3. For more details see, e.g., [EN00] and [P83], the following descriptions are mainly taken from.

Definition B.0.4 ***(Semigroup)***
Let X be a Banach space. A one parameter family $(S(t))_{t\geq 0}$ of bounded linear operators from X to X is called a semigroup on X, if the following properties hold.

i) $S(0) = I$, *where* I *is the identity operator on* X.

ii) $S(t+s) = S(t)S(s)$ *for every* $t, s \geq 0$.

Definition B.0.5 ***(Strongly continuous semigroup)***
Let X be a Banach space. The semigroup $(S(t))_{t\geq 0}$ on X is called a strongly continuous semigroup or C_0-semigroup on X, if

$$\lim_{t\searrow 0} S(t)x = x$$

for every $x \in X$.

Definition B.0.6 ***(Generator of a C_0-semigroup)***
Let X be a Banach space and $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on X. The

operator $A : D(A) \subset X \to X$ defined by

$$D(A) = \left\{ x \in X \mid \lim_{t \searrow 0} \frac{S(t)x - x}{t} \in X \right\}$$

and

$$Ax = \lim_{t \searrow 0} \frac{S(t)x - x}{t} \quad \textit{for every}\, x \in D(A)$$

is called the (infinitesimal) generator of the strongly continuous semigroup $(S(t))_{t \geq 0}$.

Proposition B.0.11 *Let X be a Banach space and A be the generator of a strongly continuous semigroup $(S(t))_{t \geq 0}$. Then $D(A)$ is dense in X and A is a closed linear operator that determines the strongly continuous semigroup uniquely. Moreover, if $x \in D(A)$, then $S(t)x \in D(A)$ and the function*

$$[0, \infty) \ni t \mapsto S(t)x \in X$$

is differentiable, which means that difference quotients have a limit in the sense of norm convergence in X. It holds,

$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax.$$

Proof: See, e.g., Section 11.1.2 in [RR93]. □

Proposition B.0.12 *Let X be a Banach space and A be the generator of a strongly continuous semigroup $(S(t))_{t \geq 0}$. Then the following assertions are equivalent.*

i) A is bounded, i.e. there exists $M > 0$ such that $\|Ax\|_X \leq M \cdot \|x\|_X$ for every $x \in D(A)$.

ii) $D(A)$ is closed in X.

iii) $D(A) = X$.

iv) $(S(t))_{t \geq 0}$ is uniformly continuous, i.e. $\lim_{t \searrow 0} \|S(t) - I\|_{\mathcal{L}(X)} = 0$.

In each case, the semigroup is given by

$$S(t) = \exp(tA) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

for every $t \geq 0$.

Proof: See, e.g., Section II.1. in [EN00]. $\square$

Definition B.0.7 ***(Abstract Cauchy problem and its classical solution)***
Let X be a Banach space, $v : [0, \infty) \to X$ be a function, $A : D(A) \subset X \to X$ be a linear operator and $v_0 \in X$. Then the initial value problem

$$\begin{aligned} \frac{d}{dt} v(t) &= Av(t), \quad t > 0, \\ v(0) &= v_0, \end{aligned} \tag{B.1}$$

is called the abstract Cauchy problem with respect to A and the initial value v_0. The function v is called a classical solution of the abstract Cauchy problem, if $v(t)$ is continuous with $v(t) \in D(A)$ for $t \geq 0$ as well as $v(t)$ is continuous differentiable for $t > 0$ and (B.1) holds.

Proposition B.0.13 *Let X be a Banach space and A be the generator of a strongly continuous semigroup $(S(t))_{t \geq 0}$. Then the function v, represented by*

$$v(t) = S(t)v_0$$

for every $t \geq 0$ and $v_0 \in D(A)$, is the unique classical solution of the abstract Cauchy problem (B.1).

Proof: See, e.g., Section II.6. in [EN00]. $\square$

Definition B.0.8 ***(Mild solution)***
Let X be a Banach space, $v_0 \in X$ and A be the generator of a strongly continuous semigroup $(S(t))_{t \geq 0}$. The function $v : [0, \infty) \to X$, given by

$$v(t) = S(t)v_0$$

is called the mild solution of the abstract Cauchy problem (B.1).

Proposition B.0.14 *Let X be a Banach space and A be the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$. Then the mild solution of the abstract Cauchy problem (B.1) exists uniquely for every $v_0 \in X$. Moreover, if $v_0 \in D(A)$, then the mild solution of (B.1) is a classical solution.*

Proof: See, e.g., Section II.6. in [EN00] and Section 11.1.3 in [RR93]. □

Example B.0.1 ***(Laplace operator as generator)***
In this example, let $d \in \mathbb{N}$ and H be the separable real Hilbert space $H = L_2\left((0,1)^d\right)$ with the norm $\|\cdot\|_H$ and the scalar product $\langle\cdot,\cdot\rangle_H$. Moreover, the linear operator

$$\Delta : D(\Delta) \subset H \to H$$

denotes the d-dimensional Laplace operator with Dirichlet boundary conditions, i.e.

$$\Delta h = \sum_{\ell=1}^{d} \frac{\partial^2}{\partial u_\ell^2} h$$

for every $h \in D(\Delta)$ with the second weak partial derivatives $\partial^2/\partial u_\ell^2$, $\ell = 1, \ldots, d$, and

$$D(\Delta) = H^2\left((0,1)^d\right) \cap H_0^1\left((0,1)^d\right).$$

In this domain, for $n \in \mathbb{N}$, $H^n\left((0,1)^d\right)$ is the Sobolev space

$$H^n\left((0,1)^d\right) = \left\{h \in L_2\left((0,1)^d\right) \mid D^\omega h \in L_2\left((0,1)^d\right) \forall \omega : |\omega|_1 \leq n\right\},$$

where $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{N}_0^d$ denotes a multi-index with $|\omega|_1 = \sum_{i=1}^d \omega_i$ and D^ω denotes the weak partial derivative with respect to ω, i.e.

$$D^\omega = \frac{\partial^{|\omega|_1}}{\partial u_1^{\omega_1} \cdots \partial u_d^{\omega_d}}.$$

Moreover, $H_0^1\left((0,1)^d\right)$ is the closure of $C_0^\infty\left((0,1)^d\right)$ in the space $H^1\left((0,1)^d\right)$, i.e.

$$H_0^1\left((0,1)^d\right) = \overline{C_0^\infty\left((0,1)^d\right)}^{H^1},$$

where $C_0^\infty\left((0,1)^d\right)$ denotes the space of infinitely many times differentiable functions with compact support in $(0,1)^d$. See, e.g., [AF03] for more details about Sobolev spaces.

Note that it holds $\Delta h_j = -\mu_j \cdot h_j$ with the orthonormal basis $(h_j)_{j\in\mathbb{N}^d} \subset D(\Delta)$ of H given by

$$h_j(u) = 2^{d/2} \cdot \prod_{\ell=1}^{d} \sin(j_\ell \cdot \pi \cdot u_\ell), \quad u \in (0,1)^d,$$

and

$$\mu_j = \pi^2 \cdot |j|_2^2$$

for every $j \in \mathbb{N}^d$ and $|\cdot|_2$ denotes the Euclidean norm. See, e.g., [RR93] for more details. Thus, by the theorem of Hille and Yosida in Section II.3. in [EN00], the Laplace operator Δ is the generator of the strongly continuous semigroup $(S(t))_{t\geq 0}$ with the expansion

$$S(t)h = \sum_{j\in\mathbb{N}^d} \exp(-\mu_j t) \cdot \langle h, h_j \rangle \cdot h_j$$

for every $h \in H$ and $t \geq 0$. $\diamond$

Appendix C

Auxiliary Results and Estimates

In this appendix we state some lemmata that are used in the proofs of Chapter 3. Remember, that the symbols $\preceq$ and $\asymp$ are introduced in Definition 3.0.1.

Lemma C.0.1 *Let $T > 0$, $f \in C^1([0,T])$ and $(\beta(t))_{t\in[0,T]}$ be a scalar Brownian motion. Then*

$$\int_0^T f(t)\,d\beta(t) = f(T)\beta(T) - \int_0^T f'(t)\beta(t)\,dt.$$

Let furthermore $\mu \geq 1$ and $y_0 \in \mathbb{R}$. Then the scalar stochastic differential equation

$$\begin{aligned} dY(t) &= -\mu Y(t)\,dt + f(t)\,d\beta(t), \quad 0 < t \leq T, \\ Y(0) &= y_0 \end{aligned}$$

has the solution

$$Y(t) = y_0 \cdot \exp(-\mu t) + \int_0^t f(s) \cdot \exp(-\mu(t-s))\,d\beta(s)$$

for $t \in [0,T]$.

Proof: See, e.g., Sections 3.2 and 4.4 in [KP92]. $\square$

Lemma C.0.2 *For $d \in \mathbb{N}$ let $B_d = \left\{x \in \mathbb{R}^d \,\middle|\, |x|_2 < 1\right\}$ be the centred open unit ball in $\mathbb{R}^d$, where $|\cdot|_2$ denotes the Euclidean norm, and let ρ_d be the $d-$dimensional Lebesgue measure. Then it holds*

$$\rho_d(B_d) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)},$$

where Γ *is the Gamma function, i.e.*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \qquad 0 < x < \infty.$$

Furthermore, for $R_1, R_2 \in [0,\infty]$ *with* $R_1 < R_2$*, let* $f : [R_1, R_2] \to \mathbb{R}$ *be a continuous function and* $K = \left\{ x \in \mathbb{R}^d \,\middle|\, R_1 \le |x|_2 \le R_2 \right\}$*. Then it holds*

$$\int_K f(|x|_2)\, dx = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{R_1}^{R_2} f(r) r^{d-1}\, dr.$$

Proof: For the proofs and more details see, e.g., [J01]. □

Lemma C.0.3 *Let* $d \in \mathbb{N}$ *and for* $R > 1$ *define* $J_R = \left\{ j \in \mathbb{N}^d \,\middle|\, 1 \le |j|_2 \le R \right\}$*. Then it holds for* $\kappa > d$*,*

$$\sum_{j \notin J_R} |j|_2^{-\kappa} \asymp \int_R^\infty r^{-\kappa+d-1}\, dr, \tag{C.1}$$

and for $\kappa \in \mathbb{R}$*,*

$$\sum_{j \in J_R} |j|_2^{-\kappa} \asymp \int_1^R r^{-\kappa+d-1}\, dr. \tag{C.2}$$

The constants, hidden in $\asymp$*, only depend on the dimension* d *and the parameter* κ*.*

Proof: From the proof of the integral convergence criterion, it is obvious that for $\kappa > d$,

$$\int_{\lfloor R \rfloor + 1}^\infty x^{-\kappa}\, dx \le \sum_{j=\lfloor R \rfloor + 1}^\infty j^{-\kappa} \le \int_{\lfloor R \rfloor}^\infty x^{-\kappa}\, dx$$

and, furthermore,

$$\int_{\left\{ x \in \mathbb{R}^d \,\middle|\, |x|_2 \ge R+1 \right\}} |x|_2^{-\kappa}\, dx \le \sum_{\left\{ j \in \mathbb{N}^d \,\middle|\, |j|_2 > R \right\}} |j|_2^{-\kappa} \le \int_{\left\{ x \in \mathbb{R}^d \,\middle|\, |x|_2 \ge R \right\}} |x|_2^{-\kappa}\, dx.$$

Since there exists a constant $c_{d,\kappa} > 0$, which may only depend on d and κ, satisfying

$$c_{d,\kappa} \cdot \int_{\left\{ x \in \mathbb{R}^d \,\middle|\, |x|_2 \ge R \right\}} |x|_2^{-\kappa}\, dx \le \int_{\left\{ x \in \mathbb{R}^d \,\middle|\, |x|_2 \ge R+1 \right\}} |x|_2^{-\kappa}\, dx,$$

we obtain

$$\sum_{\left\{j\in\mathbb{N}^d \,\middle|\, |j|_2>R\right\}} |j|_2^{-\kappa} \asymp \int_{\left\{x\in\mathbb{R}^d \,\middle|\, |x|_2\geq R\right\}} |x|_2^{-\kappa}\, dx.$$

Now we use Lemma C.0.2 with $K = \left\{x \in \mathbb{R}^d \,\middle|\, |x|_2 \geq R\right\}$ and $f(|x|_2) = |x|_2^{-\kappa}$ to get (C.1). Analogously we can derive (C.2) and the proof of the Lemma is complete. $\square$

Lemma C.0.4 *Let $d \in \mathbb{N}$, $\kappa > 1$ and for $i, j \in \mathbb{N}^d$ put*

$$\delta_{ij} = \begin{cases} \prod\limits_{\substack{\ell=1 \\ i_\ell \neq j_\ell}}^{d} \frac{1}{|i_\ell - j_\ell|}, & \textit{if} \quad i \neq j, \\ 1, & \textit{if} \quad i = j. \end{cases}$$

Then for $i \in \mathbb{N}^d$, it holds

$$\sum_{j\in\mathbb{N}^d} \delta_{ij}^{\kappa} \preceq 1$$

and

$$\sum_{j\in\mathbb{N}^d} \left(\frac{\delta_{ij}}{|j|_2}\right)^{\kappa} \preceq \left(\frac{1}{|i|_2}\right)^{\kappa}.$$

The constants, hidden in $\preceq$, only depend on the dimension d and the parameter κ.

Proof: The first estimate follows from

$$\sum_{\substack{j\in\mathbb{N}^d \\ j\neq i}} \prod_{\substack{\ell=1 \\ j_\ell\neq i_\ell}}^{d} \frac{1}{|i_\ell - j_\ell|^{\kappa}} = \prod_{\ell=1}^{d} \left(\sum_{\substack{j_\ell=1 \\ j_\ell\neq i_\ell}}^{\infty} \frac{1}{|i_\ell - j_\ell|^{\kappa}} \right) \leq \left(2\sum_{j=1}^{\infty} \frac{1}{j^{\kappa}} \right)^{d} < \infty.$$

To show the second estimate, we conclude that for every $m \in \{1,\dots,d\}$ there exist unspecified constants $c_{d,\kappa} > 0$, which only depend on d and κ, such that

$$\begin{aligned} i_m^\kappa \sum_{\substack{j\in\mathbb{N}^d \\ j\neq i}} \left(\frac{\delta_{ij}}{|j|_2}\right)^\kappa &\leq i_m^\kappa \left(\sum_{\substack{j_m=1 \\ j_m\neq i_m}}^d \frac{1}{j_m^\kappa}\frac{1}{|i_m-j_m|^\kappa}\right)\prod_{\substack{\ell=1 \\ \ell\neq m}}^d \left(\sum_{\substack{j_\ell=1 \\ j_\ell\neq i_\ell}}^d \frac{1}{|i_\ell-j_\ell|^\kappa}\right) \\ &\leq c_{d,\kappa}\sum_{\substack{j=1 \\ j\neq i_m}}^d \frac{i_m^\kappa}{j^\kappa}\frac{1}{|i_m-j|^\kappa} \\ &\leq c_{d,\kappa}\left(\sum_{j=1}^{\lceil i_m/2\rceil} j^{-\kappa} + \sum_{j=\lceil i_m/2\rceil+1}^{i_m-1} |i_m-j|^{-\kappa} + \sum_{j=i_m+1}^{\infty} |i_m-j|^{-\kappa}\right) \\ &\leq c_{d,\kappa}. \end{aligned}$$

Therefore we have

$$\sum_{j\in\mathbb{N}^d}\left(\frac{\delta_{ij}}{|j|_2}\right)^\kappa \leq \frac{c_{d,\kappa}}{d\cdot \max\limits_{m=1,\dots,d}\{i_m^\kappa\}} \preceq \frac{1}{\left(\sum\limits_{m=1}^d i_m\right)^\kappa}.$$

This finishes the proof. $\square$

Lemma C.0.5 *Let $d\in\mathbb{N}$, $\beta>1$, $\gamma\in\{0\}\cup\{x\in\mathbb{R}\mid x>d\}$ and for $i,j\in\mathbb{N}^d$ put*

$$\delta_{ij} = \begin{cases} \prod\limits_{\substack{\ell=1 \\ i_\ell\neq j_\ell}}^d \frac{1}{|i_\ell-j_\ell|}, & \textit{if} \quad i\neq j, \\ 1, & \textit{if} \quad i=j. \end{cases}$$

Then for $j\in\mathbb{N}^d$, it holds

$$\sum_{i\in\mathbb{N}^d}\left(\frac{\delta_{ij}^\beta}{|i|_2^\gamma}\right) \preceq \begin{cases} |j|_2^{-\gamma} + \prod\limits_{\ell=1}^d j_\ell^{-\gamma/d}, & \textit{if} \quad \gamma<\beta\cdot d, \\ \prod\limits_{\ell=1}^d j_\ell^{-\beta}, & \textit{if} \quad \gamma\geq\beta\cdot d. \end{cases}$$

The constant, hidden in $\preceq$, only depends on the dimension d and on the parameters β and γ.

Proof: If $\gamma = 0$, the assertion follows from Lemma C.0.4. If $\gamma > d$, the assertion is proven for $\beta = 2$ in [MGR07a], Lemma 11. Now, we follow this proof with $\beta > 1$. First, we consider $\gamma \geq \beta \cdot d$. Hence,

$$\begin{aligned}\sum_{i\in\mathbb{N}^d} |i|_2^{-\gamma} \cdot \delta_{ij}^{\beta} &\preceq \sum_{i\in\mathbb{N}^d} \left(\prod_{\ell=1}^{d} i_\ell^{-\gamma/d}\right) \cdot \delta_{ij}^{\beta} \\ &= \prod_{\ell=1}^{d} \left(\sum_{i_\ell\in\mathbb{N}} i_\ell^{-\gamma/d} \cdot \min(|i_\ell - j_\ell|^{-\beta}, 1)\right).\end{aligned}$$

Observe, that $\gamma/d \geq \beta > 1$. Thus,

$$\begin{aligned}\sum_{i_\ell\in\mathbb{N}} i_\ell^{-\gamma/d} \cdot \min(|i_\ell - j_\ell|^{-\beta}, 1) &\preceq j_\ell^{-\gamma/d} + \sum_{i_\ell \leq j_\ell/2} i_\ell^{-\gamma/d} \cdot j_\ell^{-\beta} + \sum_{\substack{i_\ell > j_\ell/2 \\ i_\ell \neq j_\ell}} j_\ell^{-\gamma/d} \cdot |i_\ell - j_\ell|^{-\beta} \\ &\preceq j_\ell^{-\beta},\end{aligned}$$

as requested. In the case $\gamma < \beta \cdot d$, we put

$$A_S = \{i \in \mathbb{N}^d \mid i_\ell = j_\ell \text{ iff } \ell \notin S\}$$

for $S \subset \{1, \ldots, d\}$ and prove

$$\sum_{i\in A_S} |i|_2^{-\gamma} \cdot \prod_{\ell\in S} |i_\ell - j_\ell|^{-\beta} \preceq |j|_2^{-\gamma} + \prod_{\ell=1}^{d} j_\ell^{-\gamma/d} \tag{C.3}$$

for every S by induction. Clearly, (C.3) holds if $S = \emptyset$. Now, we assume that $|S| = s \geq 1$ and that (C.3) holds for every proper subset of S. Without loss of generality we may assume that $S = \{1, \ldots, s\}$. Put

$$a = \left(\sum_{\ell=s+1}^{d} j_\ell^2\right)^{1/2}$$

and let

$$B = \{(i_2, \ldots, i_s) \in \mathbb{N}^{s-1} \mid i_\ell \neq j_\ell \text{ for every } \ell\},$$

if $s \geq 2$ and $B = \{0\}$ otherwise. Then

$$\sum_{i\in A_S} |i|_2^{-\gamma} \cdot \prod_{\ell\in S} |i_\ell - j_\ell|^{-\beta} = \Sigma_{\leq} + \Sigma_{>}$$

with

$$\Sigma_{\leq} = \sum_{i \in B} \sum_{i_1 \leq j_1/2} \left(i_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2} \cdot \prod_{\ell=1}^{s} |i_\ell - j_\ell|^{-\beta}$$

and

$$\Sigma_{>} = \sum_{i \in B} \sum_{\substack{i_1 > j_1/2 \\ i_1 \neq j_1}} \left(i_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2} \cdot \prod_{\ell=1}^{s} |i_\ell - j_\ell|^{-\beta}.$$

First, we derive an upper bound for the sum $\Sigma_{>}$. For every $i_1 > j_1/2$, we have

$$\left(i_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2} \preceq \left(j_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2}$$

with $i \in B$. Thus, by hypothesis, we get

$$\begin{aligned} \Sigma_{>} &\preceq \sum_{i \in B} \sum_{\substack{i_1 > j_1/2 \\ i_\ell \neq j_\ell}} \left(j_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2} \cdot \prod_{\ell=1}^{s} |i_\ell - j_\ell|^{-\beta} \\ &\preceq \sum_{i \in B} \left(j_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2} \cdot \prod_{\ell=2}^{s} |i_\ell - j_\ell|^{-\beta} \\ &\preceq |j|_2^{-\gamma} + \prod_{\ell=1}^{d} j_\ell^{-\gamma/d}. \end{aligned} \tag{C.4}$$

To derive an upper bound for the sum $\Sigma_{\leq}$, we distinguish the two cases

$$j_1^2 \leq |i|_2^2 + a^2$$

and

$$j_1^2 > |i|_2^2 + a^2.$$

In the first case, we have

$$\left(i_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2} \preceq \left(j_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2}$$

for every $i_1 \in \mathbb{N}$. Hence, similar to (C.4), we get

$$\begin{aligned}
\Sigma_{\leq} &\preceq \sum_{i \in B} \sum_{i_1 \leq j_1/2} \left(j_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2} \cdot \prod_{\ell=1}^{s} |i_\ell - j_\ell|^{-\beta} \\
&\preceq \sum_{i \in B} \left(j_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2} \cdot \prod_{\ell=2}^{s} |i_\ell - j_\ell|^{-\beta} \\
&\preceq |j|_2^{-\gamma} + \prod_{\ell=1}^{d} j_\ell^{-\gamma/d}. \qquad \text{(C.5)}
\end{aligned}$$

In the second case, we use

$$\left(i_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2} \preceq \left(\prod_{\ell=1}^{s} i_\ell^{-\gamma/d}\right) \cdot \left(\prod_{\ell=s+1}^{d} j_\ell^{-\gamma/d}\right)$$

to obtain

$$\left(i_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2} \preceq \left(j_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2} + \left(\prod_{\ell=1}^{s} i_\ell^{-\gamma/d}\right) \cdot \left(\prod_{\ell=s+1}^{d} j_\ell^{-\gamma/d}\right).$$

for every $i \in B$. Furthermore,

$$\sum_{i \in B} \sum_{i_1 \leq j_1/2} \left(\prod_{\ell=1}^{s} i_\ell^{-\gamma/d}\right) \cdot \left(\prod_{\ell=s+1}^{d} j_\ell^{-\gamma/d}\right) \cdot \prod_{\ell=1}^{s} |i_\ell - j_\ell|^{-\beta} \preceq j_1^{-\beta} \cdot \prod_{\ell=2}^{d} j_\ell^{-\gamma/d}.$$

Note that $-\beta < -\gamma/d$ and we therefore get

$$\begin{aligned}
\Sigma_{\leq} &\preceq \sum_{i \in B} \sum_{i_1 \leq j_1/2} \left(j_1^2 + |i|_2^2 + a^2\right)^{-\gamma/2} \cdot \prod_{\ell=1}^{s} |i_\ell - j_\ell|^{-\beta} + \prod_{\ell=1}^{d} j_\ell^{-\gamma/d} \\
&\preceq |j|_2^{-\gamma} + \prod_{\ell=1}^{d} j_\ell^{-\gamma/d}
\end{aligned}$$

analogously to (C.5), which finishes the proof. $\square$

Lemma C.0.6 *Let $n \in \mathbb{N}$, $T > 0$ and $\mu \geq 1$ with $n \geq \max(\mu, T)$. Then there exists a constant $c_T > 0$, which only depends on the parameter T, such that*

$$\sum_{k=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \left(\exp(-\mu(T-t)) - \left(1 + \mu \cdot \frac{T}{n}\right)^{-(n-k)} \right)^2 dt \leq c_T \cdot \frac{\mu}{n^2}. \tag{C.6}$$

Proof: We prove (C.6) by extending the integrand,

$$\begin{aligned}
&\sum_{k=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \left(\exp(-\mu(T-t)) - \left(1 + \mu \cdot \frac{T}{n}\right)^{-(n-k)} \right)^2 dt \\
\leq\ & 2 \left(\sum_{k=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \left(\exp(-\mu(T-t)) - \exp\left(-\mu(T - \frac{k}{n}T)\right) \right)^2 dt \right. \\
& \left. + \sum_{k=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \left(\exp\left(-\mu(T - \frac{k}{n}T)\right) - \left(1 + \mu \cdot \frac{T}{n}\right)^{-(n-k)} \right)^2 dt \right).
\end{aligned}$$

Using the mean value theorem, we obtain for the first series

$$\begin{aligned}
&\sum_{k=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \left(\exp(-\mu(T-t)) - \exp\left(-\mu(T - \frac{k}{n}T)\right) \right)^2 dt \\
=\ & \exp(-2\mu T) \sum_{k=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \left(\exp(\mu t) - \exp\left(\mu \frac{k}{n} T\right) \right)^2 dt \\
\leq\ & \exp(-2\mu T) \sum_{k=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \left(\left(\mu t - \mu \frac{k}{n} T\right)^2 \exp\left(2\mu \frac{k+1}{n} T\right) \right) dt \\
=\ & \frac{1}{3} \mu^2 \frac{T^3}{n^3} \exp(-2\mu T) \sum_{k=0}^{n-1} \exp\left(2\mu \frac{k+1}{n} T\right)
\end{aligned}$$

and, thus, by $\mu \leq n$

$$\begin{aligned}
&\sum_{k=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \left(\exp(-\mu(T-t)) - \exp\left(-\mu(T-\frac{k}{n}T)\right)\right)^2 dt \\
&\leq \mu^2 \frac{T^2}{n^2} \exp\left(2\frac{\mu}{n}T\right) \exp(-2\mu T) \frac{T}{n} \sum_{k=0}^{n-1} \exp\left(2\mu\frac{k}{n}T\right) \\
&\leq \exp(2T) T^2 \frac{\mu^2}{n^2} \exp(-2\mu T) \int_0^T \exp(2\mu x)\, dx \\
&= \frac{1}{2} \exp(2T) T^2 \frac{\mu}{n^2} (1 - \exp(-2\mu T)) \\
&\leq \exp(2T) T^2 \frac{\mu}{n^2}.
\end{aligned}$$

To estimate the second series, we use that for any $x, y \in \mathbb{R}$ and $m \in \mathbb{N}$ holds

$$x^m - y^m = (x-y) \sum_{\ell=0}^{m-1} x^\ell y^{m-\ell-1},$$

which can be shown by induction. Hence,

$$\begin{aligned}
&\sum_{k=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \left(\exp\left(-\mu(T-\frac{k}{n}T)\right) - \left(1+\mu\cdot\frac{T}{n}\right)^{-(n-k)}\right)^2 dt \\
&= \frac{T}{n} \sum_{k=0}^{n-1} \left(\left(\exp(\mu\frac{T}{n})\right)^{-(n-k)} - \left(1+\mu\cdot\frac{T}{n}\right)^{-(n-k)}\right)^2 \\
&= \frac{T}{n} \sum_{k=0}^{n-1} \left(\left(\exp(-\mu\frac{T}{n}) - \left(1+\mu\cdot\frac{T}{n}\right)^{-1}\right)\right. \\
&\qquad \times \left.\sum_{\ell=0}^{n-k-1} \left(\exp(-\mu\frac{T}{n}(n-k-\ell-1)) \left(1+\mu\cdot\frac{T}{n}\right)^{-\ell}\right)\right)^2.
\end{aligned}$$

Because of

$$\begin{aligned}
0 \leq (1+x)^{-1} - \exp(-x) &= (1+x)^{-1} \int_0^x y \cdot \exp(-y)\, dy \\
&\leq \frac{1}{2}(1+x)^{-1} x^2 \qquad \text{(C.7)}
\end{aligned}$$

for $x \geq 0$ and

$$(1+x)^{-1} \leq \exp\left(-\frac{x}{2}\right) \tag{C.8}$$

for $0 \leq x \leq 1$, it follows that

$$\begin{aligned}
&\sum_{k=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \left(\exp\left(-\mu(T-\frac{k}{n}T)\right) - \left(1+\mu\cdot\frac{T}{n}\right)^{-(n-k)} \right)^2 dt \\
&\leq \frac{1}{4}\frac{T}{n}\left(1+\mu\frac{T}{n}\right)^{-2} \mu^4 \frac{T^4}{n^4} \sum_{k=0}^{n-1} \left(\sum_{\ell=0}^{n-k-1} \exp\left(-\mu\frac{T}{2n}(n-k-1)\right)\right)^2 \\
&\leq \mu^4 \frac{T^5}{n^5} \sum_{k=0}^{n-1} \left((n-k)^2 \exp\left(-\mu\frac{T}{n}(n-k-1)\right)\right) \\
&\leq \mu^4 \frac{T^5}{n^5} \exp(T) \sum_{k=0}^{n-1} \left((n-k-1)^2 \exp\left(-\mu\frac{T}{n}(n-k)\right)\right) \\
&\quad + \mu^4 \frac{T^5}{n^5} \exp(T) \sum_{k=0}^{n-1} \left((2(n-k)-1) \exp\left(-\mu\frac{T}{n}(n-k)\right)\right) \\
&\leq \mu^4 \frac{T^5}{n^5} \exp(T) \sum_{k=0}^{n-1} \left(k^2 \exp\left(-\mu\frac{k+1}{n}T\right)\right) \\
&\quad + 2\mu^4 \frac{T^5}{n^5} \exp(T) \sum_{k=0}^{n-1} \left((k+1) \exp\left(-\mu\frac{k+1}{n}T\right)\right) \\
&\leq 2\mu^4 \frac{T^4}{n^4} \exp(T) \left(\int_0^T x^2 \exp(-\mu x)\,dx + \int_0^T x\exp(-\mu x)\,dx + \int_0^T \exp(-\mu x)\,dx \right) \\
&\leq 2\exp(T) T^4 \frac{\mu^2}{n^2} \left(\frac{2}{\mu^3} + \frac{1}{\mu^2} + \frac{1}{\mu} \right) \\
&\leq 4\exp(T) T^4 \frac{\mu}{n^2},
\end{aligned}$$

which finishes the proof of (C.6). □

Lemma C.0.7 *Let $n \in \mathbb{N}$, $T > 0$ and $\mu \geq 1$. Let $(t_k)_{k\in\{0,\ldots,n\}}$ be a sequence of regular time nodes in $[0,T]$ w.r.t. the density $\psi(t) = \exp(-\mu/3\cdot(T-t))$, $t \in [0,T]$, i.e.*

$$\int_0^{t_k} \exp(-\mu/3\cdot(T-t))\,dt = \frac{k}{n}\int_0^T \exp(-\mu/3\cdot(T-t))\,dt, \qquad k = 0,\ldots,n. \tag{C.9}$$

Then there exist two positive constants c_1 and c_2, such that

$$\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (\exp(-\mu(T-t)) - \exp(-\mu(T-t_k)))^2 \, dt \le c_1 \cdot \frac{1}{\mu n^2} \tag{C.10}$$

and

$$\sum_{k=0}^{n-1} \left(\exp(-\mu(T-t_k)) - \prod_{\ell=k}^{n-1} (1+\mu(t_{\ell+1}-t_\ell))^{-1} \right)^2 (t_{k+1}-t_k) \le c_2 \cdot \frac{1}{\mu n^2}. \tag{C.11}$$

Proof: First, we proof the estimate (C.10). For this purpose, we note that for $t \in [t_k, t_{k+1}]$ by (C.9) it holds that

$$\begin{aligned}
&\exp(-\mu(T-t)) - \exp(-\mu(T-t_k)) \\
&= \left(\exp(-\frac{2\mu}{3}(T-t)) + \exp(-\frac{\mu}{3}(T-t) - \frac{\mu}{3}(T-t_k)) + \exp(-\frac{2\mu}{3}(T-t_k)) \right) \\
&\quad \times \left(\exp(-\frac{\mu}{3}(T-t)) - \exp(-\frac{\mu}{3}(T-t_k)) \right) \\
&\le 3\exp(-\frac{2\mu}{3}(T-t)) \cdot \left(\exp(-\frac{\mu}{3}(T-t_{k+1})) - \exp(-\frac{\mu}{3}(T-t_k)) \right) \\
&= 3\exp(-\frac{2\mu}{3}(T-t)) \cdot \frac{\mu}{3} \int_{t_k}^{t_{k+1}} \exp(-\frac{\mu}{3}(T-s)) \, ds \\
&= \exp(-\frac{2\mu}{3}(T-t)) \cdot \frac{\mu}{n} \int_0^T \exp(-\frac{\mu}{3}(T-s)) \, ds \\
&\le \frac{3}{n} \exp(-\frac{2\mu}{3}(T-t)) \cdot \left(1 - \exp(-\frac{\mu}{3}T) \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (\exp(-\mu(T-t)) - \exp(-\mu(T-t_k)))^2 \, dt &\le \frac{9}{n^2} \int_0^T \exp(-\frac{4\mu}{3}(T-t)) \, dt \\
&\le \frac{27}{4} \frac{1}{\mu n^2}.
\end{aligned}$$

To prove (C.11), we put $\Delta_k = t_{k+1} - t_k$ for $k = 0, \dots, n-1$ and note that

$$\begin{aligned}
\Delta_k &\le \exp(\frac{\mu}{3}(T-t_k)) \int_{t_k}^{t_{k+1}} \exp(-\frac{\mu}{3}(T-t)) \, dt \\
&\le \frac{3}{\mu n} \exp(\frac{\mu}{3}(T-t_k)), \quad k = 0, \dots, n-1,
\end{aligned} \tag{C.12}$$

such that with (C.7) we have for $k = 0, \dots, n-1$,

$$\begin{aligned}(1+\mu\Delta_k)^{-1} - \exp(-\mu\Delta_k) &\leq \frac{1}{2}(1+\mu\Delta_k)^{-1}(\mu\Delta_k)^2 \\ &\leq \frac{9}{2n^2}(1+\mu\Delta_k)^{-1}\exp(\frac{2\mu}{3}(T-t_k)). \qquad \text{(C.13)}\end{aligned}$$

Now, we follow the proof of Lemma 3 in [MGRW07] and put

$$\delta_k = \left(\prod_{\ell=k}^{n-1}(1+\mu\Delta_\ell)^{-1}\right) - \exp(-\mu(T-t_k))$$

for $k = 0, \dots, n-1$ and $\delta_n = 0$ to obtain with (C.13)

$$\begin{aligned}\delta_k &= (1+\mu\Delta_k)^{-1}\delta_{k+1} + \exp(-\mu(T-t_{k+1}))\left((1+\mu\Delta_k)^{-1} - \exp(-\mu\Delta_k)\right) \qquad \text{(C.14)} \\ &\leq (1+\mu\Delta_k)^{-1}\left(\delta_{k+1} + \frac{9}{2n^2}\exp(-\frac{\mu}{3}(T-t_k))\exp(\mu\Delta_k)\right), \quad k = 0, \dots, n-1.\end{aligned}$$

Furthermore, we use that the nodes $(t_k)_{k\in\{0,\dots,n\}}$, defined by (C.9), satisfy

$$t_k = \frac{3}{\mu}\log\left(\frac{k}{n}(\exp(\frac{\mu}{3}T)-1)+1\right), \quad k = 0, \dots, n,$$

and therefore

$$\mu\Delta_k \leq 3\log 2, \quad k = 1, \dots, n-1.$$

Thus, by (C.8),

$$\begin{aligned}\delta_k &\leq \left(1+\frac{\mu\Delta_k}{3\log 2}\right)^{-1}\left(\delta_{k+1} + \frac{36}{n^2}\exp(-\frac{\mu}{3}(T-t_k))\right) \\ &\leq \delta_{k+1}\exp(-\frac{\mu\Delta_k}{6\log 2}) + \frac{36}{n^2}\exp(-\frac{\mu}{6\log 2}(T-t_k)), \quad k = 1, \dots, n-1.\end{aligned}$$

Accordingly, by induction,

$$\begin{aligned}\delta_k &\leq \frac{36}{n^2}(n-k)\exp(-\frac{\mu}{6\log 2}(T-t_k)) \\ &\leq \frac{36}{n}\exp(-\frac{\mu}{6\log 2}(T-t_k)), \quad k = 1, \dots, n-1. \qquad \text{(C.15)}\end{aligned}$$

To derive δ_0, we remember from (C.14) that

$$\delta_0 = (1+\mu\Delta_0)^{-1}\delta_1 + \exp(-\mu(T-t_1))\left((1+\mu\Delta_0)^{-1} - \exp(-\mu\Delta_0)\right).$$

The second summand in this equation can be estimated with

$$0 < \sup_{t\geq 0} t\cdot \exp(-2/3\cdot t) \leq 1$$

as follows

$$\begin{aligned}
&\exp(-\mu(T-t_1))\left((1+\mu\Delta_0)^{-1} - \exp(-\mu\Delta_0)\right) \\
&= \exp(-\mu(T-t_1))(1+\mu\Delta_0)^{-1}\int_0^{\mu\Delta_0} t\cdot\exp(-t)\,dt \\
&\leq \exp(-\frac{\mu}{3}(T-t_1))(1+\mu\Delta_0)^{-1}\int_0^{\mu\Delta_0}\exp(-t/3)\,dt \\
&= (1+\mu\Delta_0)^{-1}\mu\int_0^{t_1}\exp(-\frac{\mu}{3}(T-t))\,dt \\
&= (1+\mu\Delta_0)^{-1}\frac{\mu}{n}\int_0^{T}\exp(-\frac{\mu}{3}(T-t))\,dt \\
&\leq \frac{3}{n}(1+\mu\Delta_0)^{-1},
\end{aligned}$$

such that with (C.15) we have

$$\delta_0 \leq (1+\mu\Delta_0)^{-1}\left(\frac{36}{n}+\frac{3}{n}\right) = \frac{39}{n}(1+\mu\Delta_0)^{-1}. \tag{C.16}$$

Hence, with (C.16), (C.15) and (C.12),

$$\begin{aligned}
\sum_{k=0}^{n-1}\delta_k^2\Delta_k &= \delta_0^2\Delta_0 + \sum_{k=1}^{n-1}\delta_k^2\Delta_k \\
&\leq \frac{39^2}{n^2}(1+\mu\Delta_0)^{-2}\Delta_0 + \sum_{k=1}^{n-1}\frac{36^2}{n^2}\exp(-\frac{\mu}{3\log 2}(T-t_k))\frac{3}{\mu n}\exp(\frac{\mu}{3}(T-t_k)) \\
&\leq \frac{39^2}{n^2}\frac{\Delta_0}{1+\mu\Delta_0} + \frac{3\cdot 36^2}{\mu n^3}\sum_{k=1}^{n-1}\exp(-\frac{\mu}{3}(T-t_k))\exp(\frac{\mu}{3}(T-t_k)) \\
&\leq 5409\frac{1}{\mu n^2},
\end{aligned}$$

which finishes the proof. □

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